LEARNING MATERIALS

ON

ENGINEERING MATHEMATICS

PREPARED BY

ANIRUDHA PANDA

16. MATRICES

1. INTRODUCTION

A rectangular array of m x n numbers (real or complex) in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as m × n matrix. Such an array is enclosed by [ ] or ( ) or || ||. An m × n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In brief, the above matrix is represented by $A = [a_{ij}]_{m\times n}$. The number a_{11}, a_{12}, ... etc., are known as the elements of the matrix A, where a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{th}$ element of the matrix $A = [a_{ij}]$.

2. ORDER OF A MATRIX

A matrix which has m rows and n columns is called a matrix of order m × n E.g. the order of $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ matrix is 2 × 3.

Note: (a) The difference between a determinant and a matrix is that a determinant has a certain value, while the matrix has none. The matrix is just an arrangement of certain quantities.

(a) The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, then the matrix is called a real matrix.

(a) An m × n matrix has m.n elements.

Illustration 1: Construct a 3×4 matrix $A = [a_{ij}]$, whose elements are given by $a_{ij} = 2i + 3j$. **(JEE MAIN)**

Sol: In this problem, i and j are the number of rows and columns respectively. By substituting the respective values of rows and columns in $a_{ij} = 2i + 3j$ we can construct the required matrix.

We have $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$; $\therefore a_{11} = 2 \times 1 + 3 \times 1 = 5$; $a_{12} = 2 \times 1 + 3 \times 2 = 8$

Similarly, $a_{13} = 11$, $a_{14} = 14$, $a_{21} = 7$, $a_{22} = 10$, $a_{23} = 13$, $a_{24} = 16$, $a_{31} = 9$, $a_{32} = 12$, $a_{33} = 15$, $a_{34} = 18$

$$\therefore A = \begin{bmatrix} 5 & 8 & 11 & 14 \\ 7 & 10 & 13 & 16 \\ 9 & 12 & 15 & 18 \end{bmatrix}$$

Illustration 2: Construct a 3 × 4 matrix, whose elements are given by: $a_{ij} = \frac{1}{2}|-3i + j|$ **(JEE MAIN)**

Sol: Method for solving this problem is the same as in the above problem.

Since $a_{ij} = \frac{1}{2}|-3i + j|$ we have

$a_{11} = \frac{1}{2}|-3(1) + 1| = \frac{1}{2}|-3 + 1| = \frac{1}{2}|-2| = \frac{2}{2} = 1$

$a_{12} = \frac{1}{2}|-3(1) + 2| = \frac{1}{2}|-3 + 2| = \frac{1}{2}|-1| = \frac{1}{2}$

$a_{13} = \frac{1}{2}|-3(1) + 3| = \frac{1}{2}|-3 + 3| = \frac{1}{2}(0) = 0$

$a_{14} = \frac{1}{2}|-3(1) + 4| = \frac{1}{2}|-3 + 4| = \frac{1}{2}$; $a_{21} = \frac{1}{2}|-3(2) + 1| = \frac{1}{2}|-6 + 1| = \frac{5}{2}$

$a_{22} = \frac{1}{2}|-3(2) + 2| = \frac{1}{2}|-6 + 2| = \frac{4}{2} = 2$; $a_{23} = \frac{1}{2}|-3(2) + 3| = \frac{1}{2}|-6 + 3| = \frac{3}{2}$

$a_{24} = \frac{1}{2}|-3(2) + 4| = \frac{1}{2}|-6 + 4| = \frac{2}{2} = 1$; Similarly $a_{31} = 4$, $a_{32} = \frac{7}{2}$, $a_{33} = 3$, $a_{34} = \frac{5}{2}$

Hence, the required matrix is given by $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

3. TYPES OF MATRICES

Row Matrix

A matrix having only one row is called a row matrix. Thus $A = [a_{ij}]_{m\times n}$ is a row matrix if m = 1;

E.g. A = [1 2 4 5] is row matrix of order 1 × 4.

Column Matrix

A matrix having only one column is called a column matrix. Thus $A = [a_{ij}]_{m\times n}$ is a column matrix if n = 1; E.g. $A = \begin{bmatrix} -1 \\ 2 \\ -4 \\ 5 \end{bmatrix}$ is column matrix of order 4 × 1.

Zero or Null Matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by 0.Thus, $A = [a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j; E.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 × 3.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3 × 2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3 × 3 null matrix.

Singleton Matrix

If in a matrix there is only one element then it is called singleton matrix. Thus, $A = [a_{ij}]_{m\times n}$ is a singleton matrix if m = n = 1. E.g. [2], [3], [a], [–3] are singleton matrices.

Horizontal Matrix

A matrix of order m × n is a horizontal matrix if n > m; E.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

Vertical Matrix

A matrix of order m × n is a vertical matrix if m > n; E.g. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

Square Matrix

If the number of rows and the number of columns in a matrix are equal, then it is called a square matrix.

Thus, $A = [a_{ij}]_{m\times n}$ is a square matrix if m = n; E.g. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3 × 3.

The sum of the diagonal elements in a square matrix A is called the trace of matrix A, and which is denoted by tr(A); $\quad tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots\ldots a_{nn}.$

Diagonal Matrix

If all the elements, except the principal diagonal, in a square matrix are zero, it is called a diagonal matrix.

Thus, a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$; E.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3 × 3, which can also be denoted by diagonal [2 3 4].

Scalar Matrix

If all the elements in the diagonal of a diagonal matrix are equal, it is called a scalar matrix. Thus, a square matrix A $= [a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0, & i \neq j \\ k, & i = j \end{cases}$ where k is a constant.

E.g. $\begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$ is a scalar Matrix.

Unit Matrix

If all the elements of a principal diagonal in a diagonal matrix are 1, then it is called a unit matrix. A unit matrix of order n is denoted by I_n. Thus, a square matrix $A = [a_{ij}]_{n\times n}$ is a unit matrix if

$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ $\qquad$ E.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: Every unit matrix is a scalar matrix.

Triangular Matrix

A square matrix is said to be a triangular matrix if the elements above or below the principal diagonal are zero. There are two types:

Upper Triangular Matrix

A square matrix $[a_{ij}]$ is called an upper triangular matrix, If $a_{ij} = 0$, when $i > j$.

E.g. $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper uriangular matrix of order 3×3.

Lower Triangular Matrix

A square matrix is called a lower triangular matrix, if $a_{ij} = 0$ when $i < j$.

E.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3×3.

Singular Matrix

Matrix A is said to be a singular matrix if its determinant $|A| = 0$, otherwise a non-singular matrix, i.e.

If $\det |A| = 0 \Rightarrow$ Singular and $\det |A| \neq 0 \Rightarrow$ non-singular

MASTERJEE CONCEPTS

- A triangular matrix $A = [a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ij} = 0\ \forall\ i=j$
- The multiplication of two triangular matrices is a triangular matrix.
- Every row matrix is also a horizontal matrix but not the converse. Similarly every column matrix is also a vertical matrix but not the converse.

Vaibhav Gupta (JEE 2009 AIR 54)

Symmetric and Skew Symmetric Matrices

Symmetric Matrix: A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$, for all i,j values;

E.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{pmatrix}$ is symmetric, because $a_{12} = 2 = a_{21}$, $a_{31} = 3 = a_{13}$ etc.

Note: A is symmetric $\Leftrightarrow A = A'$ (where A′ is the transpose of matrix)

Skew-Symmetric Matrix: A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$, for all values of i,j.

$\because a_{ij} = -a_{ji}$, for all i,j. $\Rightarrow a_{ii} = -a_{ii}$, [putting j = i] $\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$

Thus, in a skew-symmetric matrix all diagonal elements are zero; E.g. $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Note: A square matrix A is a skew-symmetric matrix $\Leftrightarrow A' = -A$.

Few results:

(a) If A is any square matrix, then A + A′ is a symmetric matrix and A – A′ is a skew-symmetric matrix.

(b) Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix. $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}(B + C)$, where B is symmetric and C is a skew symmetric matrix.

(c) If A and B are symmetric matrices, then AB is symmetric ⇔ AB = BA, i.e. A & B commute.

(d) The matrix B′AB is symmetric or skew-symmetric in correspondence if A is symmetric or skew-symmetric.

(e) All positive integral powers of a symmetric matrix are symmetric.

(f) Positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.

MASTERJEE CONCEPTS

Elements of the main diagonal of a skew-symmetric matrix are zero because by definition $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ or $a_{ii} = 0$ for all values of i.

Trace of a skew symmetric matrix is always 0. The sum of symmetric matrices is symmetric.

Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}(B + C)$, where B is symmetric and C is a skew symmetric matrix.

If A and B are symmetric matrices, then AB is symmetric ⇔ AB = BA, i.e. A & B commute. The matrix B′AB is symmetric or skew-symmetric accordingly when A is symmetric or skew symmetric. All positive integral powers of a symmetric matrix are symmetric. Positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.

Chen Reddy Sandeep Reddy (JEE 2012 AIR 62)

Hermitian and Skew-Hermitian Matrices

A square matrix $A = [a_{ij}]$ is said to be a Hermitian matrix if $a_{ij} = \overline{a_{ji}}\ \forall$ i, j; i.e. $A = A^{\theta}$

E.g. $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$ are Hermitian matrices

Note: (a) If A is a Hermitian matrix then $a_{ii} = \overline{a_{ii}} \Rightarrow a_{ii}$ is real ∀ i, thus every diagonal element of a Hermitian matrix must be real.

(b) If a Hermitian matrix over the set of real numbers is actually a real symmetric matrix; and A a square matrix, $A = [a_{ij}]$ is said to be a skew-Hermitian if $a_{ij} = -\overline{a_{ji}}$, ∀ i, j;

i.e. $A^{\theta} = -A$; E.g. $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}, \begin{pmatrix} 3i & -3+2i & -1-i \\ 3-2i & -2i & -2-4i \\ 1+i & 2+4i & 0 \end{pmatrix}$ are skew-Hermitian matrices.

(c) If A is a skew-Hermitian matrix then $a_{ii} = -\overline{a_{ii}} \Rightarrow a_{ii} + \overline{a_{ii}} = 0$

i.e. a_{ii} must be purely imaginary or zero.

(d) A skew-Hermitian matrix over the set of real numbers is actually is real skew-symmetric matrix.

4. TRACE OF A MATRIX

Let A = $[a_{ij}]_{n\times n}$ and B = $[b_{ij}]_{n\times n}$ and λ be a scalar,

(i) $tr(\lambda A) = \lambda\, tr(A)$ (ii) $tr(A + B) = tr(A) + tr(B)$ (iii) $tr(AB) = tr(BA)$

- Square Matrix
  - Triangular Matrix
    - Upper triangular Matrix
      If $a_{ij} = 0\ \forall\ i > j$
      $A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$,
    - Lower triangular Matrix
      If $a_{ij} = 0\ \forall\ i < j$
      $A = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$
  - Diagonal Matrix
    Atleast one, $a_{ii} \neq 0$ and $a_{ij} = 0$ if $i \neq j$
    $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$
    Abbreviated as dia $(d_1, d_2, d_3, \ldots\ldots d_n)$
    - Scalar Matrix
      If $d_1 = d_2 = d_3 \ldots = a \neq 0$
      $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$
    - Unit Matrix
      If $d_1 = d_2 = d_3 \ldots = 1$
      $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

5. TRANSPOSE OF A MATRIX

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called the transpose of matrix A and is denoted by A^T or A′. From the definition it is obvious that if the order of A is m × n, then the order of A^T becomes n × m; E.g. transpose of matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{2\times 3} \text{ is } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3\times 2}$$

Properties of Transpose of Matrix

(i) $(A^T)^T = A$ (ii) $(A \pm B)^T = A^T \pm B^T$ (iii) $(AB)^T = B^TA^T$ (iv) $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3 \ldots\ldots A_{n-1}A_n)^T = A_n^T A_{n-1}^T \ldots\ldots A_3^T A_2^T A_1^T$ (vi) $I^T = I$ (vii) $tr(A) = tr(A^T)$

Illustration 3: If A = $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$ then prove that $(AB)^T = B^TA^T$. **(JEE MAIN)**

Sol: By obtaining the transpose of AB i.e. $(AB)^T$ and multiplying B^T and A^T we can easily get the result.

Here, $AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1(1)-2(-1)+3(2) & 1(3)-2(0)+3(4) \\ -4(1)+2(-1)+5(2) & -4(3)+2(0)+5(4) \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 4 & 8 \end{bmatrix}$

$\therefore (AB)^T = \begin{bmatrix} 9 & 4 \\ 15 & 8 \end{bmatrix}$; $B^TA^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1(1)-1(-2)+2(3) & 1(-4)-1(2)+2(5) \\ 3(1)+0(-2)+4(3) & 3(-4)+0(2)+4(5) \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 15 & 8 \end{bmatrix} = (AB)^T$

Illustration 4: If A = $\begin{bmatrix} 5 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix}$ then what is (AB′)′ is equal to? **(JEE MAIN)**

Sol: In this problem, we use the properties of the transpose of matrix to get the required result.

We have $(AB')' = (B')'\,A' = BA' = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 18 & 7 \end{bmatrix}$

Illustration 5: If the matrix A = $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is a singular matrix then find x. Verify whether $AA^T = I$ for that value of x. **(JEE ADVANCED)**

Sol: Using the condition of singular matrix, i.e. | A | = 0, we get the value of x and then substituting the value of x in matrix A and multiplying it to its transpose we will obtain the required result.

Here, A is a singular matrix if | A | = 0, i.e., $\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$

or $\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ 0 & -x & -x \end{vmatrix} = 0$, using $R_3 \to R_3 + R_2$ or $\begin{vmatrix} 3-x & 0 & 2 \\ 2 & 3-x & 1 \\ 0 & 0 & -x \end{vmatrix} = 0$, using $C_2 \to C_2 - C_3$

or $-x(3-x)^2 = 0$, $\therefore$ x = 0, 3.

When x = 0, A = $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$; $\therefore AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & -4 \\ 2 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 17 & 16 & -16 \\ 16 & 21 & -21 \\ -16 & -21 & 21 \end{bmatrix} \neq I$

When x = 3,A = $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}$ $\therefore AA^T = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & -4 \\ 2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 4 & -16 \\ 4 & 6 & -12 \\ -16 & -12 & 36 \end{bmatrix} \neq I$

Note: A simple way to solve is that if A is a singular matrix then |A| = 0 and $|A^T| = 0$. But |I| is 1. Hence, $AA^T \neq I$ if |A| = 0.

Illustration 6: If the matrix A = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c, are positive real numbers such that abc = 1 and $A^TA = I$ then find the value of $a^3 + b^3 + c^3$. **(JEE ADVANCED)**

Sol: Here, A = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. So, $A^T = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, interchanging rows and columns.

$$\therefore A^TA = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}^2 = A^2 \therefore |A^TA| = |A^2|; \text{ But } A^TA = I \text{ (given)}. \therefore |I| = |A|^2 \Rightarrow 1 = |A|^2$$

$$\text{Now, } |A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}, R_1 \to R_1 + R_2 + R_3$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}, \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$= (a + b + c) \{(c - b)(b - c) - (a - b)(a - c)\}$ $\quad = (a + b + c)(-b^2 - c^2 + 2bc - a^2 + ac + ab - bc)$

$= -(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$ $\quad = -(a^3 + b^3 + c^3 - 3abc)$

$= -(a^3 + b^3 + c^3 - 3)$ $(\because abc = 1)$ $\quad \therefore |A|^2 = 1 \Rightarrow (a^3 + b^3 + c^3 - 3)^2 = 1$ ... (i)

As a, b, c, are positive, $\dfrac{a^3+b^3+c^3}{3} > \sqrt[3]{a^3b^3c^3}$ $(\because abc = 1)$; $\quad \therefore a^3 + b^3 + c^3 > 3$

$\therefore$ (i) $\Rightarrow a^3 + b^3 + c^3 - 3 = 1$ $\quad \therefore a^3 + b^3 + c^3 = 4$

6. MATRIX OPERATIONS

Equality of Matrices

Two matrices A and B are said to be equal if they are of the same order and their corresponding elements are equal, i.e. Two matrices $A = [a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{r\times s}$ are equal if

(a) m = r i.e. the number of rows in A = the number of rows in B.

(b) n = s, i.e. the number of columns in A = the number of columns in B

(c) $a_{ij} = b_{ij}$, for i = 1, 2, ..., m and j = 1, 2, ..., n, i.e. the corresponding elements are equal;

E.g. Matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not equal because their orders are not the same.

E.g. If $A = \begin{bmatrix} 1 & 6 & 3 \\ 5 & 2 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ are equal matrices then,

$a_1 = 1, a_2 = 6, a_3 = 3, b_1 = 5, b_2 = 2, b_3 = 1.$

Addition of Matrices

If $A[a_{ij}]_{m\times n}$ and $B[b_{ij}]_{m\times n}$ are two matrices of the same order then their sum A + B is a matrix, and each element of that matrix is the sum of the corresponding elements. i.e. $A + B = [a_{ij} + b_{ij}]_{m\times n}$

Properties of Matrix Addition: If A, B and C are matrices of same order, then

(a) A + B = B + A (Commutative law),

(b) (A + B) + C = A + (B + C) (Associative law),

(c) A + O = O + A = A, where O is zero matrix which is additive identity of the matrix,

(d) A + (–A) = 0 = (–A) + A, where (–A) is obtained by changing the sign of every element of A which is additive inverse of the matrix,

(e) $\left.\begin{array}{l} A+B=A+C \\ B \end{array}\right\} \Rightarrow B = C$

(f) tr (A ± B) = tr (A) ± tr (B)

(g) **Additive Inverse:** If A + B = 0 = B + A, then B is called additive inverse of A and also A is called the additive inverse of A.

(h) **Existence of Additive Identity:** Let $A = [a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

Subtraction of Matrices

If A and B are two matrices of the same order, then we define A – B = A + (– B).

Scalar Multiplication of Matrices

If $A=[a_{ij}]_{m\times n}$ is a matrix and k any number, then the matrix which is obtained by multiplying the elements of A by k is called the scalar multiplication of A by k and it is denoted by k A thus if $A = [a_{ij}]_{m\times n}$

Then $kA_{m\times n} = A_{m\times n}k = [ka_{i\times j}]$

Properties of Scalar Multiplication: If A, B are matrices of the same order and λ, μ are any two scalars then

(a) $\lambda (A + B) = \lambda A + \lambda B$ (b) $(\lambda + \mu)A = \lambda A + \mu A$ (c) $\lambda(\mu A) = (\lambda \mu A) = \mu(\lambda A)$

(d) $(-\lambda A) = -(\lambda A) = \lambda(-A)$ (e) tr (kA) = k tr (A)

Multiplication of Matrices

If A and B be any two matrices, then their product AB will be defined only when the number of columns in A is equal to the number of rows in B. If $A[a_{ij}]_{m\times n}$ and $B[b_{ij}]_{n\times p}$ then their product $AB = C = [c_{ij}]$, will be a matrix of order

$m \times p$, where $(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$

Proof: Let $A=[a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the $m \times p$ matrix $C = [c_{ij}]$ is called the product if $C_{ij}=A_iB_j$ Where A_i is the i^{th} row of A and B_j is the j^{th} column of B. Thus the product AB is obtained as following:

$$A = m \times n \qquad\qquad B = n \times p$$

$$\begin{array}{l} R_1 \rightarrow \\ \\ R_2 \rightarrow \\ R_i \rightarrow \\ \\ R_m \rightarrow \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \times \begin{array}{c} \begin{array}{ccccccc} C_1 & C_2 & & C_j & & C_p \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{ip} \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & \dots\dots & R_1C_j & R_1C_p \\ R_2C_1 & R_2C_2 & \dots\dots & R_2C_j & R_2C_p \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ R_iC_1 & R_iC_2 & \dots\dots & R_iC_j & R_iC_p \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ R_mC_1 & R_mC_2 & \dots\dots & R_mC_j & R_mC_p \end{bmatrix} \text{ Thus } (AB)_{ij} = A_iB_j$$

$$(AB)_{ij} = [a_{i1}\ a_{i2} \dots\dots a_{ij} \dots\dots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \dots \\ b_{ij} \\ \dots\dots \\ b_{rj} \end{bmatrix} = [a_{i1} b_{1j} + a_{i2} b_{2j} + \dots\dots + a_{in} b_{nj}]\ ; \qquad (AB)_{ij} = \sum_{r=1} (a_{ir}.b_{rj})$$

Properties of matrix multiplication:

(a) Matrix multiplication is not commutative in general, i.e. in general AB ≠ BA.

(b) Matrix multiplication is associative, i.e. (AB)C = A(BC).

(c) Matrix multiplication is distributive over matrix addition, i.e. A.(B + C) = A.B + A.C and (A + B)C = AC + BC.

(d) If A is an m × n matrix, then $I_mA = A = AI_n$.

(e) The product of two matrices can be a null matrix while neither of them is null, i.e. if AB = 0, it is not necessary that either A = 0 or B = 0.

(f) If A is an m × n matrix and O is a null matrix then $A_{m\times n}.\ O_{n\times p} = O_{m\times p}$, i.e. the product of the matrix with a null matrix is always a null matrix.

(g) If AB = 0 (It does not mean that A = 0 or B = 0, again the product of two non-zero matrices may be a zero matrix).

(h) If AB = AC $\Rightarrow$ B $\neq$ C (Cancellation Law is not applicable).

(i) tr(AB) = tr(BA).

Illustration 7: If $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$ find AB and BA if possible **(JEE MAIN)**

Sol: Using matrix multiplication. Here, A is a 3 × 3 matrix and B is a 3 × 2 matrix, therefore, A and B are conformable for the product AB and it is of the order 3 × 2 such that

$$(AB)_{11} = \text{(First row of A) (First column of B)} = [2\ 1\ 3] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 2 \times 1 + 1 \times 2 + 3 \times 4 = 16$$

$$(AB)_{12} = \text{(First row of A) (Second column of B)} = [2\ 1\ 3] \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = 2 \times (-2) + 1 \times 1 + 3 \times (-3) = -12$$

$(AB)_{21}$ = (Second row of A) (First column of B)= $[3\ -2\ 1]\begin{bmatrix}1\\2\\4\end{bmatrix} = 3\times 1+(-2)\times 2+1\times 4 = 3$

Similarly $(AB)_{22} = -11$, $(AB)_{31} = 3$ and $(AB)_{32} = -1$; $\therefore$ AB = $\begin{bmatrix}16 & -12\\ 3 & -11\\ 3 & -1\end{bmatrix}$

BA is not possible since number of columns of B ≠ number of rows of A.

Illustration 8: Find the value of x and y if $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$ **(JEE MAIN)**

Sol: Using the method of multiplication and addition of matrices, then equating the corresponding elements of L.H.S. and R.H.S., we can easily get the required values of x and y.

We have, $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix} \Rightarrow \begin{bmatrix}2 & 6\\0 & 2x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix} \Rightarrow \begin{bmatrix}2+y & 6+0\\0+1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$

Equating the corresponding elements, a_{11} and a_{22}, we get

$2 + y = 5 \Rightarrow y = 3$; $2x + 2 = 8 \Rightarrow 2x = 6 \Rightarrow x = 3$;

Hence x = 3 and y = 3.

Illustration 9: Find the value of a, b, c and d, if $\begin{bmatrix}a-b & 2a+c\\2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\0 & 13\end{bmatrix}$ **(JEE MAIN)**

Sol: As the two matrices are equal, their corresponding elements are equal. Therefore, by equating the corresponding elements of given matrices we will obtain the value of a, b, c and d.

$\begin{bmatrix}a-b & 2a+c\\2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\0 & 13\end{bmatrix}$ (given)

$a - b = -1$ ... (i)

$2a + c = 5$ ... (ii)

$2a - b = 0$ ... (iii)

$3c + d = 13$ ... (iv)

Subtracting equation (i) from (iii), we have a = 1;

Putting the value of a in equation (i), we have $1 - b = -1 \Rightarrow b = 2$;

Putting the value of a in equation (ii), we have $2 + c = 5 \Rightarrow c = 3$;

Putting the value of c in equation (iv), we find $9 + d = 13 \Rightarrow d =$

Hence a = 1, b = 2, c = 3, d = 4.

Illustration 10: Find x and y, if $2x + 3y = \begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}$ and $3x + 2y = \begin{bmatrix}2 & -2\\-1 & 5\end{bmatrix}$ **(JEE MAIN)**

Sol: Solving the given equations simultaneously, we will obtain the values of x and y.

We have $2x + 3y = \begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}$ ... (i)

$3x + 2y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$ ... (ii)

Multiplying (i) by 3 and (ii) by 2, we get $6x + 9y = \begin{bmatrix} 6 & 9 \\ 12 & 0 \end{bmatrix}$ ... (iii)

$6x + 4y = \begin{bmatrix} 4 & -4 \\ -2 & 10 \end{bmatrix}$ ... (iv)

Subtracting (iv) from (iii), we get $5y = \begin{bmatrix} 6-4 & 9+4 \\ 12+2 & 0-10 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix}$

$\Rightarrow y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & \frac{-10}{5} \end{bmatrix} \Rightarrow y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

Putting the value of y in (iii), we get $2x + 3\begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$

$\Rightarrow 2x = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix} = \begin{bmatrix} 2-\frac{6}{5} & 3-\frac{39}{5} \\ 4-\frac{42}{5} & 0+6 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{24}{5} \\ -\frac{22}{5} & 6 \end{bmatrix} \Rightarrow x = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$

Hence $x = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$ and $y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

Illustration 11: If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+z \\ 2b+4 & -21 & 0 \end{bmatrix}$ then find the values of a, b, c, x, y and z.

(JEE ADVANCED)

Sol: As the two matrices are equal, their corresponding elements are also equal. Therefore, by equating the corresponding elements of the given matrices, we will obtain the values of a, b, c, x, y and z.

$$\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+z \\ 2b+4 & -21 & 0 \end{bmatrix}$$

Comparing both sides, we get $x + 3 = 0 \quad \Rightarrow x = -3$ ... (i)

and $z + 4 = 6 \quad \Rightarrow z = 6 - 4 \quad \Rightarrow z = 2$ ... (ii)

and $2y - 7 = 3y - 2 \Rightarrow 2y - 3y = -2 + 7 \quad \Rightarrow -y = 5 \quad \Rightarrow y = -5$ ... (iii)

and $a - 1 = -3 \quad \Rightarrow a = -3 + 1 \quad \Rightarrow a = -2$ ... (iv)

and $b - 3 = 2b + 4 \Rightarrow b - 2b = 4 + 3 \quad \Rightarrow -b = 7 \quad \Rightarrow b = -7$ ... (v)

and $2c + 2 = 0 \quad \Rightarrow 2c + 2 = 0 \quad \Rightarrow 2c = -2 \quad \Rightarrow c = \frac{-2}{2} \quad \Rightarrow c = -1$ ... (vi)

[from(2)] Thus; $a = -2$, $b = -7$, $c = -1$, $x = -3$, $y = -5$ and $z = 2$

7. RANK OF A MATRIX

If $A = (a_{ij})_{m\times n}$ is a matrix, and B is its sub-matrix of order r, then | B |, the determinant is called r-rowed minor of A.

Definition: Let $A = (a_{ij})_{m\times n}$ be a matrix. A positive integer r is said to be a rank of A if

(a) A possesses at least one r-rowed minor which is different from zero; and

(b) Every (r + 1) rowed minor of A is zero.

From (ii), it automatically follows that all minors of higher order are zeros. We denote rank of A by $\rho(A)$

Note: The rank of a matrix does not change when the following elementary row operations are applied to the matrix:

(a) Two rows are interchanged ($R_i \leftrightarrow R_j$);

(b) A row is multiplied by a non-zero constant, ($R_i \to kR_i$, with $k \neq 0$);

(c) A constant multiple of another row is added to a given row ($R_i \to R_i, + kR_j$) where $i \neq j$.

Note: The arrow $\to$ means "replaced by".

Note that the application of these elementary row operations does not change a singular matrix to a non-singular matrix nor does a non-singular matrix change to a singular matrix. Therefore, the order of the largest non-singular square sub-matrix is not affected by the application of any of the elementary row operations. Thus, the rank of a matrix does not change by the application of any of the elementary row operations. A matrix obtained from a given matrix by applying any of the elementary row operations is said to be equivalent to it. If A and B are two equivalent matrices, we write A ~ B. Note that if A ~ B, then $\rho(A) = \rho(B)$

By using the elementary row operations, we shall try to transform the given matrix in the following

form $\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ \cdots & \cdots & \cdots & \cdots \\ & & & \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & 0\ldots & * \end{pmatrix}$

Where * stands for zero or non-zero element. That is, we shall try to make a_{ii} as 1 and all the elements below a_{ij} as zero.

MASTERJEE CONCEPTS

A non zero matrix A is said to have rank r, if

- Every square sub-matrix of order (r + 1) or more is singular;
- There exists at least one square sub-matrix or order r which is non singular.

B Rajiv Reddy (JEE 2012 AIR 11)

Illustration 12: For what values of x does the matrix $\begin{bmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{bmatrix}$ have the rank 2? **(JEE ADVANCED)**

Sol: The given matrix has only one 3rd-order minor. In order that the rank arrive at 2, we must bring about its determinant to zero. Hence, by applying the invariance method we can obtain values of x.

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0 \qquad \text{... (i)}$$

Now, using $R_1 \to R_1 - R_3$

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = \begin{vmatrix} 1+x & 0 & -1-x \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix}; \text{ using } C_3 \to C_3 + C_1 = \begin{vmatrix} 1+x & 0 & 0 \\ 1 & 7+x & 7 \\ 2 & 5 & 5+x \end{vmatrix}$$

$$= (1+x)\begin{vmatrix} 7+x & 7 \\ 5 & 5+x \end{vmatrix} = (1+x)\,[(7+x(5+x) - 35] = (1+x)(x^2+12x) = x(1+x)(x+12)$$

$\therefore$ (i) holds for $x = 0, -1, -12$

When $x = 0$, the matrix $= \begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 6 \\ 2 & 5 & 3 \end{bmatrix}$ Clearly, a minor $\begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix} \neq 0$, So, the rank = 2

When $x = -1$, the matrix $= \begin{bmatrix} 2 & 5 & 2 \\ 1 & 6 & 6 \\ 2 & 5 & 2 \end{bmatrix}$ Clearly, a minor $\begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} \neq 0$, So, the rank = 2

When $x = -12$, the matrix $= \begin{bmatrix} -9 & 5 & 2 \\ 1 & -5 & 6 \\ 2 & 5 & -9 \end{bmatrix}$ Clearly, a minor $\begin{bmatrix} -9 & 5 \\ 1 & -5 \end{bmatrix} \neq 0$, So, the rank = 2

$\therefore$ The matrix has the rank 2 if $x = 0, -1, -12$.

8. POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then, $A^2 = A.A$; $A^3 = A.A.A = A^2A$. Also for any positive integers m, n:

(a) $A^mA^n = A^{m+n}$ **(b)** $(A^m)^n = A^{mn} = (A^n)^m$

(c) $I^n = I$, $I^m = I$ **(d)** $A^0 = I_n$

Matrix polynomial: If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_nx^0$, then we define a matrix polynomial a,b

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots + a_nI_n$ where A is the given square matrix. If f(A) is a null matrix, then A is called the zero or root of the matrix polynomial f(A)

9. SPECIAL MATRICES

(a) **Idempotent Matrix:** A square matrix is idempotent, provided $A^2 = A$. For an idempotent matrix A, $A^n = A\ \forall\ n>2, n \in N \Rightarrow A^n = A, n \geq 2$.

For an idempotent matrix A, det A = 0 or 1 A^2, $|A|^2 = |A|$).

(b) **Nilpotent Matrix:** A nilpotent matrix is said to be nilpotent of index p, $(p \in N)$, if $A^p = O$, $A^{p-1} \neq O$, i.e. if p is the least positive integer for which $A^p = O$, then A is said to be nilpotent of index p.

(c) Periodic Matrix: A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K, then A is periodic with period K, i.e. if K is the least positive integer for which $A^{K+1} = A$, and A is said to be periodic with period K. If K =1 then A is called idempotent.

E.g. the matrix $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ has the period 1.

Note: (i) Period of a square null matrix is not defined. (ii) Period of an idempotent matrix is 1.

(d) Involutary Matrix: If $A^2 = I$, the matrix is said to be an involutary matrix. An involutary matrix its own inverse

E.g. (i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

MASTERJEE CONCEPTS

Two matrices cannot be added if they are of different order

If A is an involutary matrix, then $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $(I + A)(I - A) = 0$

Chinmay S Purandare (JEE 2012 AIR 698)

Illustration 13: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix}$ and $f(x) = x^2 - 5x + 6I_3$. Find f(A). **(JEE MAIN)**

Sol: By using methods of multiplication and addition of matrices we will obtain the required result. Here $f(A) = A^2 - 5A + 6I_3$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix}^2 - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ -5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2+0\times2+1(-1) & 2\times0+0\times1+1(-1) & 2\times1+0\times3+1\times0 \\ 2\times2+1\times2+3(-1) & 2\times0+1\times1+3(-1) & 2\times1+1\times3+3\times0 \\ (-1)2+(-1)2+0(-1) & (-1)0+(-1)1+0(-1) & (-1)1+(-1)3+0\times0 \end{bmatrix} + \begin{bmatrix} 6-10 & 0-0 & 0-5 \\ 0-10 & 6-5 & 0-15 \\ 0-(-5) & 0-(-15) & 6-0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 3 & -2 & 5 \\ -4 & -1 & -4 \end{bmatrix} + \begin{bmatrix} -4 & 0 & -5 \\ -10 & 1 & -15 \\ 5 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3-4 & -1+0 & 2-5 \\ 3-10 & -2+1 & 5-15 \\ -4+5 & -1+5 & -4+6 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -7 & -1 & -10 \\ 1 & 4 & 2 \end{bmatrix}$$

Illustration 14: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be such that $A^3 = 0$, but $A \neq 0$, then **(JEE MAIN)**

Sol: (a) As $A^3 = 0$, we get $|A^3| = 0$; $|A^3| = 0 \Rightarrow |A| = 0 \Rightarrow ad - bc = 0$

In this problem, $A^3 = 0$ means | A | also is equal to 0; therefore, by calculating A^2 we can obtain the result.

(a) $A^2 = 0$ (b) $A^2 = A$ (c) $A^2 = I - A$ (d) None of these

Also, $A^2 = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & bc+d^2 \end{pmatrix} = \begin{pmatrix} a^2+ad & (a+d)b \\ (a+d)c & ad+d^2 \end{pmatrix} = (a + d)\,A$

If a + d = 0, we get $A^2 = 0$. But, if $a + d \neq 0$, then $A^3 = A^2A = (a + d)\,A^2 \Rightarrow 0 = (a + d)\,A^2 \Rightarrow A^2 = 0$

Illustration 15: If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then which one of the following holds for all $n \geq 1$, by the principle of mathematical induction. **(JEE MAIN)**

(a) $A^n = nA + (n - 1)\,I$ (b) $A^n = 2^{n-1} A + (n - 1)\,I$

(c) $A^n = nA - (n - 1)\,I$ (d) $A^n = 2^{n-1} A - (n - 1)\,I$

Sol: By substituting n = 2 we can determine the correct answer.

For n = 2, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ For n = 2, RHS of (a) = $2A + I = 3\begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix} \neq A^2$

For n = 2, RHS of (b) = $2A + I \neq A^2$ So possible answer is (c) or (d)

In fact $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ which equals $nA - (n - 1)\,I$;

Alternatively. Write A = I + B Where $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

As $B^2 = 0$, we get $B^r = 0\ \forall r \geq 2$

By the binomial theorem, $A^n = I + nB = I + n(A - I) = nA - (n - 1)I$

10. ADJOINT OF A MATRIX

Let the determinant of a square matrix A be | A |

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The matrix formed by the cofactors of the elements in | A | is $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Where $A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,a_{33} - a_{23}.\,a_{32}$

$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -a_{21}.\,a_{33} + a_{23}.\,a_{31}$; $A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{22}\,a_{31}$;

$A_{21} = (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -a_{12}\,a_{33} + a_{13}.\,a_{32}$; $A_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}\,a_{33} - a_{13}.\,a_{31}$;

$A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -a_{11}\, a_{32} + a_{12}.\, a_{31}$; $A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}\, a_{23} - a_{13}.\, a_{22}$;

$A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -a_{11}\, a_{23} + a_{13}\, a_{21}$; $A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}.\, a_{21}$;

Then the transpose of the matrix of co-factors is called the adjoint of the matrix A and is written as adj A.

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

The product of a matrix A and its adjoint is equal to unit matrix multiplied by the determinant A.

Let A be a square matrix, then (Adjoint A). A = A. (Adjoint A) = | A |. I

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$$A.\,(\text{adj. } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I.$$

Illustration 16: If $A = \begin{bmatrix} x & 3 & 2 \\ -3 & y & -7 \\ -2 & 7 & 0 \end{bmatrix}$ and A = – A′, then x + y is equal to

(a) 2 (b) –1 (c) 0 (d) 12 **(JEE MAIN)**

Sol: (c) A = – A′; ⇔ A is skew-symmetric matrix; ⇒ diagonal elements of A are zeros

⇒ x = 0, y = 0; ∴ x + y = 0

Illustration 17: If A and B are two skew-symmetric matrices of order n, then, **(JEE MAIN)**

(a) AB is a skew-symmetric matrix (b) AB is a symmetric matrix

(c) AB is a symmetric matrix if A and B commute (d) None of these

Sol: (c) We are given A′ = – A and B′ = – B; Now, (AB)′ = B′A′ = (–B) (–A) = BA = AB if A and B commute.

Illustration 18: Let A and B be two matrices such that AB′ + BA′ = O. If A is skew symmetric ,then BA **(JEE MAIN)**

(a) Symmetric (b) Skew symmetric (c) Invertible (d) None of these

Sol: (c) we have, (BA)′ = A′B′ = – AB′ [∵ A is skew symmetric]; = BA′ = B(–A) = – BA ⇒ BA is skew symmetric.

Illustration 19: Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$, then the co-factors of elements of A are given by - **(JEE MAIN)**

Sol: Co-factors of the elements of any matrix are obtain by eliminating all the elements of the same row and column and calculating the determinant of the remaining elements.

$A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 3 \times 3 - 4 \times 4 = -7$

$A_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$, $A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$; $A_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6$, $A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2$, $A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$; $A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$, $A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

$\therefore$ Adj A = $\begin{vmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix}$

Illustration 20: Which of the following statements are false – **(JEE MAIN)**

(a) If | A | = 0, then | adj A | = 0;

(b) Adjoint of a diagonal matrix of order 3 × 3 is a diagonal matrix;

(c) Product of two upper triangular matrices is a upper triangular matrix;

(d) adj (AB) = adj (A) adj (B);

Sol: (d) We have, adj (AB) = adj (B) adj (A) and not adj (AB) = adj (A) adj (B)

11. INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that AB = BA = I (I = unit matrix)

Then B is called the inverse of A, i.e. B = A^{-1} and A is the inverse of B. Condition for a square matrix A to possess an inverse is that the matrix A is non-singular, i.e., | A | ≠ 0. If A is a square matrix and B is its inverse then AB = I. Taking determinant of both sides | AB | = | I | or | A | | B | = I. From this relation it is clear that | A | ≠ 0, i.e. the matrix A is non-singular.

To find the inverse of matrix by using adjoint matrix:

We know that, A. (Adj A) = | A | I or $\frac{A.(AdjA)}{|A|} = I$ (Provided | A | ≠ 0)

and A. A^{-1} = I; $\therefore$ $A^{-1} = \frac{1}{|A|}$ (Adj. A)

Illustration 21: Let A = $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$. What is inverse of A ? **(JEE MAIN)**

Sol: By using the formula $A^{-1} = \frac{\text{adj}\,A}{|A|}$ we can obtain the value of A^{-1}.

We have $A_{11} = \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = 2 \quad A_{12} = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = 21$

And similarly $A_{13} = -18, A_{31} = 4, A_{32} = -8, A_{33} = 4, A_{21} = +6, A_{22} = -7, A_{23} = 6$

$\therefore \text{adj } A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix} \qquad \text{Also } |A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = \{4\times(-7) - (-6)\times 5 - 3\times(-6)\}$

$= -28 + 30 + 18 = 20 \qquad \therefore \quad A^{-1} = \dfrac{\text{adj} A}{|A|} = \dfrac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Illustration 22: If the product of a matrix A and $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ is the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, then A^{-1} is given by: **(JEE MAIN)**

(a) $\begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ -2 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$ (d) None of these

Sol: (a) We know if AB = C, then $B^{-1}A^{-1} = C^{-1} \Rightarrow A^{-1} = BC^{-1}$ by using this formula we will get value of A^{-1} in the above problem.

Here, $A\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$

Illustration 23: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Prove that $(AB)^{-1} = B^{-1}A^{-1}$. **(JEE ADVANCED)**

Sol: By obtaining | AB | and adj AB we can obtain $(AB)^{-1}$ by using the formula $(AB)^{-1} = \dfrac{\text{adj} AB}{|AB|}$. Similarly we can also obtain the values of B^{-1} and A^{-1}.Then by multiplying B^{-1} and A^{-1} we can prove the given problem.

$$\text{Here, } AB = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2+1 & 4+3-1 & 10+1-1 \\ 0+2+0 & 0+3+0 & 0+1+0 \\ 1+6+1 & 2+9-1 & 5+3-1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 10 \\ 2 & 3 & 1 \\ 8 & 10 & 7 \end{bmatrix}$$

$$\text{Now, } |AB| = \begin{vmatrix} 5 & 6 & 10 \\ 2 & 3 & 1 \\ 8 & 10 & 7 \end{vmatrix} = 5(21-10) - 6(14-8) + 10(20-24) = 55 - 36 - 40 = -21.$$

$$\text{The matrix of cofactors of } |AB| \text{ is } = \begin{bmatrix} 3(7)-1(10) & -\{2(7)-8(1)\} & 2(10)-3(8) \\ -\{6(7)-10(10)\} & 5(7)-8(10) & -\{5(10-6(8)\} \\ 6(1)-10(3) & -\{5(1)-2(10)\} & 5(3)-6(2) \end{bmatrix} = \begin{bmatrix} 11 & -6 & -4 \\ 58 & -45 & -2 \\ -24 & 15 & 3 \end{bmatrix}$$

$$\therefore \text{adj } AB = \begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}; \qquad \therefore (AB)^{-1} = \frac{\text{adj} AB}{|AB|} = \frac{-1}{21}\begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}$$

Next, $|B| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(3-1) - 2(2+1) + 5(2+3) = 21$

$\therefore B^{-1} = \dfrac{\text{adj } B}{|B|} = \dfrac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$; $\quad |A| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{vmatrix} = 1(-2+1) = -1$

$\therefore A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{-1}\begin{bmatrix} -1 & -2 & 1 \\ 0 & -1 & 0 \\ -1 & -5 & 2 \end{bmatrix}$; $\quad \therefore B^{-1}A^{-1} = -\dfrac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 \\ 0 & -1 & 0 \\ -1 & -5 & 2 \end{bmatrix}$

$= -\dfrac{1}{21}\begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}$ $\quad$ Thus, $(AB)^{-1} = B^{-1}A^{-1}$

Illustration 24: If $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$, then **(JEE ADVANCED)**

(a) $x = \pm 1/\sqrt{6}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$ (b) $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$

(c) $x = \pm 1/\sqrt{6}$, $y = \pm 1/\sqrt{2}$, $z = \pm 1/\sqrt{3}$ (d) $x = \pm 1/\sqrt{2}$, $y = \pm 1/3$, $z = \pm 1/\sqrt{2}$

Sol: (b) Given that $A' = A^{-1}$ and we know that $AA^{-1} = I$ and therefore $AA' = I$. Using the multiplication method we can obtain values of x, y and z.

$A' = A^{-1} \Leftrightarrow AA' = I$

Now, $AA' = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 4y^2+z^2 & 2y^2-z^2 & -2y^2+z^2 \\ 2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ -2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix}$

Thus, $AA' = I$ $\quad \Rightarrow 4y^2 + z^2 = 1,\ 2y^2 - z^2 = 0,\quad x^2 + y^2 + z^2 = 1,\ x^2 - y^2 - z^2 = 0$

$\therefore \quad x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$

Illustration 25: If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then **(JEE ADVANCED)**

(a) $x = 1, y = -1$ (b) $x = -1, y = 1$

(c) $x = 2, y = -1/2$ (d) $x = 1/2, y = \dfrac{1}{2}$

Sol: (a) We know $AA^{-1} = I$, hence by solving it we can obtain the values of x and y.

We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & y+1 \\ 0 & 1 & 2(y+1) \\ 4(1-x) & 3(x-1) & 2+xy \end{bmatrix}$$

$\Rightarrow \quad 1 - x = 0,\ x - 1 = 0;\ \ y + 1 = 0,\ y + 1 = 0,\ 2 + xy = 1; \qquad \therefore \quad x = 1, y = -1$

12. SYSTEM OF LINEAR EQUATIONS

Let the equations be

$a_1x + a_2y + a_3z = d_1$

$b_1x + b_2y + b_3z = d_2$

$c_1x + c_2y + c_3z = d_3$

We write the above equations in the matrix form as follows

$$\begin{bmatrix} a_1x+a_2y+a_3z \\ b_1x+b_2y+b_3z \\ c_1x+c_2y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad \Rightarrow AX = B \qquad \ldots \text{(i)}$$

Where, $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Multiplying (i) by A^{-1}, we get $A^{-1}AX = A^{-1}B \Rightarrow I.\,X = A^{-1}B \Rightarrow X = A^{-1}B$

Solution to a System of Equations

A set of values of x, y, z which simultaneously satisfy all the equations is called a solution to the system of equations.

Consider, $x + y + z = 9 \qquad 2x - y + z = 5 \qquad 4x + y - z = 7$

Here, the set of values – $x = 2, y = 3, z = 4$, is a solution to the system of linear equations.

Because, $2 + 3 + 4 = 9 \qquad 4 - 3 + 4 = 5 \qquad 8 + 3 - 4 = 7$

Consistent Equations

If the system of equations has one or more solution, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations. For example, the system of linear equations $x + 3y = 5$ $x - y = 1$ is consistent, because $x = 2, y = 1$ is a solution to it. However, the system of linear equations $x + 3y = 5$ $2x + 6y = 8$ is inconsistent, because there is no set of values of x and y which may satisfy the two equations simultaneously.

Condition for consistency of a system of linear equation AX = B

(a) If $|A| \neq 0$, then the system is consistent and has a unique solution, given by $X = A^{-1}B$

(b) If $|A| = 0$, and $(\text{Adj } A)\, B \neq 0$ then the system is inconsistent.

(c) If $|A| = 0$, and $(\text{Adj } A)\, B = 0$, then the system is consistent and has infinitely many solutions.

Note, $AX = 0$ is known as homogeneous system of linear equations, here $B = 0$. A system of homogeneous equations is always consistent.

The system has non-trivial solution (non-zero solution), if $|A| = 0$

Theorem 1: Let AX = B be a system of linear equations, where A is the coefficient matrix. If A is invertible then the system has a unique solution, given by $X = A^{-1}B$

Proof: $AX = B$; Multiplying both sides by A^{-1}. Since A^{-1} exists $\Rightarrow |A| \neq 0$

$\Rightarrow \quad A^{-1}AX = A^{-1}B \qquad \Rightarrow \quad IX = A^{-1}B \qquad \Rightarrow \quad X = A^{-1}B$

Thus, the system of equations AX = B has a solution given by $X = A^{-1}B$

Uniqueness: If AX = B has two sets of solutions X_1 and X_2, then

$AX_1 = B$ and $AX_2 = B$ (Each equal to B) $\Rightarrow \quad AX_1 = AX_2$

By cancellation law, A being invertible $\Rightarrow$ $X_1 = X_2$

Hence, the given system AX = B has a unique solution. **Proved**

Note: A homogeneous system of equations is always consistent.

Illustration 26: Let $A = \begin{bmatrix} x+y & y \\ 2x & x-y \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. If AB = C. Then find the matrix A^2 **(JEE MAIN)**

Sol: By solving AB = C we get the values of x and y. Then by substituting these values in A we obtain A^2.

Here $\begin{bmatrix} x+y & y \\ 2x & x-y \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2(x+y)-y \\ 2x.2-(x-y) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow 2(x + y) - y = 3$ and $4x - (x - y) = 2$

$\Rightarrow 2x + y = 3$ and $3x + y = 2$ Subtracting the two equations, we get, $x = -1$, So, $y = 5$.

$\therefore A = \begin{bmatrix} -1+5 & 5 \\ 2(-1) & -1-5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & -6 \end{bmatrix}$; $\therefore A^2 = \begin{bmatrix} 4 & 5 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -2 & -6 \end{bmatrix}$

$= \begin{bmatrix} 4\times4+5(-2) & 4\times5+5(-6) \\ -2\times4+(-6)(-2) & -2\times5+(-6)(-6) \end{bmatrix}$ $= \begin{bmatrix} 6 & -10 \\ 4 & 26 \end{bmatrix}$

Illustration 27: Solve the following equations by matrix inversion

$2x + y + 2z = 0$ $2x - y + z = 10$ $x + 3y - z = 5$ **(JEE ADVANCED)**

Sol: The given equation can be written in a matrix form as AX = D and then by obtaining A^{-1} and multiplying it on both sides we can solve the given problem.

$\begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$ $\therefore$ AX = D where $A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$\Rightarrow A^{-1}(AX) = A^{-1}D \Rightarrow (A^{-1}A)X = A^{-1}D$ $\Rightarrow IX = A^{-1}D \Rightarrow X = A^{-1}D$ ...(i)

Now $A^{-1} = \frac{\text{adj}\,A}{|A|}$; $|A| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(1-3) - 1(-2-1) + 2(6+1) = 13$

The matrix of co-factors of | A | is $\begin{bmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & 2 & -4 \end{bmatrix}$. So, adj $A = \begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$. $\therefore$ from (i), $X = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 0+70+15 \\ 0-40+10 \\ 0-50-20 \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}$; $\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix} \Rightarrow x = \frac{85}{13}, y = \frac{-30}{13}, z = \frac{-70}{13}$

Illustration 28: If $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then matrix A equals:

(a) $\begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 1 \\ 34 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 3 \\ 13 & 8 \end{bmatrix}$ **(JEE ADVANCED)**

Sol: (a) We know that if XAY = I, then $A = X^{-1} Y^{-1} = (YX)^{-1}$.

In this case $YX = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}$; $\therefore A = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$

Illustration 29: The system of equations $\begin{pmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 3 \\ -1 \end{pmatrix}$ has no solution if a and b are

(a) $a = -3, b \neq 1/3$ (b) $a = 2/3, b \neq 1/3$

(c) $a \neq 1/4, b = 1/3$ (d) $a \neq -3, b \neq 1/3$ **(JEE ADVANCED)**

Sol: By applying row operation in the given matrices and comparing them we can obtain the required result.

(a) The augmented matrix is given by $(A|B) = \left(\begin{array}{ccc|c} 3 & -2 & 1 & b \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{array}\right)$

Applying $R_1 \to 2R_1 - R_2$, we get $(A|B) \sim \left(\begin{array}{ccc|c} 1 & 4 & -7 & 2b-3 \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{array}\right)$

Applying $R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$, we get $(A|B) \sim \left(\begin{array}{ccc|c} 1 & 4 & -7 & 2b-3 \\ 0 & -28 & 44 & 18-10b \\ 0 & -7 & a+14 & 5-4b \end{array}\right)$

The system of equations will have no solution if $\dfrac{-28}{-7} = \dfrac{44}{a+14} \neq \dfrac{18-10b}{5-4b}$

$\Rightarrow a + 14 = 11$ and $20 - 16b \neq 18 - 10b$

$\Rightarrow a = -3$ and $b \neq -1/3$.

Illustration 30: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ equals:

(a) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ **(JEE ADVANCED)**

Sol: (c) Adding Au_1 and Au_2 we get $A(u_1 + u_2)$. Then using the invariance method we obtain $u_1 + u_2$.

By adding, we have $A(u_1 + u_2) = Au_1 + Au_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

We then solve the above equation for $u_1 + u_2$, if we consider the augmented matrix $(A|B) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right)$

Applying $R_3 \to R_3 - 2R_2 + R_1$ and $R_2 \to R_2 - 2R_1$, we get $(A|B) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow u_1 + u_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

PROBLEM-SOLVING TACTICS

If A, B are square matrices of order n, and I_n is a corresponding unit matrix, then

(a) $A(\text{adj}.A) = |A| I_n = (\text{adj } A) A$

(b) $|\text{adj } A| = |A|^{n-1}$ (Thus A (adj A) is always a scalar matrix)

(c) $\text{adj } (\text{adj}.A) = |A|^{n-2} A$

(d) $|\text{adj } (\text{adj}.A)| = |A|^{(n-1)^2}$

(e) $\text{adj } (A^T) = (\text{adj } A)^T$

(f) $\text{adj } (AB) = (\text{adj } B)(\text{adj } A)$

(g) $\text{adj } (A^m) = (\text{adj } A)^m, m \in N$

(h) $\text{adj } (kA) = k^{n-1} (\text{adj}. A), k \in R$

(i) $\text{adj } (I_n) = I_n$

(j) **)** adj 0 = 0

(k) A is symmetric $\Rightarrow$ adj A is also symmetric

(l) A is diagonal $\Rightarrow$ adj A is also diagonal

(m) A is triangular $\Rightarrow$ adj A is also triangular

(n) A is singular $\Rightarrow |\text{adj } A| = 0$

FORMULAE SHEET

(a) Types of matrix:

(i) Symmetric Matrix: A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$, for all i,j.

(ii) Skew-Symmetric Matrix: when $a_{ij} = -a_{ji}$

(iii) Hermitian and skew – Hermitian Matrix: $A = A^{\theta}$ (Hermitian matrix)

$A^{\theta} = -A$ (skew-Hermitian matrix)

(iv) Orthogonal matrix: if $AA^T = I_n = A^TA$

(v) Idempotent matrix: if $A^2 = A$

(vi) Involuntary matrix: if $A^2 = I$ or $A^{-1} = A$

(vii) Nilpotent matrix: if $\exists\ p \in N$ such that $A^P = 0$

(b) Trace of matrix:

(i) $tr(\lambda A) = \lambda\ tr(A)$

(ii) $tr(A + B) = tr(A) + tr(B)$

(iii) $tr(AB) = tr(BA)$

(c) Transpose of matrix:

(i) $(A^T)^T = A$ **(ii)** $(A \pm B)^T = A^T \pm B^T$ **(iii)** $(AB)^T = B^TA^T$ **(iv)** $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3 \ldots\ldots A_{n-1}A_n)^T = A_n^T A_{n-1}^T \ldots\ldots A_3^T A_2^T A_1^T$ **(vi)** $I^T = I$ **(vii)** $tr(A) = tr(A^T)$

(d) Properties of multiplication:

(i) $AB \neq BA$ **(ii)** $(AB)C = A(BC)$ **(iii)** $A.(B + C) = A.B + A.C$

(e) Adjoint of a Matrix:

(i) $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$ **(ii)** $|\text{adj } A| = |A|^{n-1}$

(iii) $(\text{adj } AB) = (\text{adj } B)(\text{adj } A)$ **(iv)** $\text{adj }(\text{adj } A) = |A|^{n-2}$

(v) $(\text{adj } KA) = K^{n-1}(\text{adj } A)$

(e) Inverse of a matrix: A^{-1} exists if A is non singular i.e. $|A| \neq 0$

(i) $A^{-1} = \frac{1}{|A|}$ (Adj. A) **(ii)** $A^{-1}A = I_n = AA^{-1}$

(iii) $(A^T)^{-1} = (A^{-1})^T$ **(iv)** $(A^{-1})^{-1} = A$

(v) $|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$