LECTURE NOTES ON

ENGINEERING MATHEMATICS-III

3rd SEMESTER ETC

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Matrices

Basic concepts of matrixs :-

- A matrix is a rectangular ~~arrange~~ can arrangement of numbers either real or complex or both.
- A matrix with m rows & n columns is called m×n matrix or m by n.
- A rectangular arrange of elements of the ~~for~~ form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_m & am_2 & \cdots\cdots & a_{mn} \end{bmatrix}$$

(m) one rows & (n) one columns.

we can also write it as $[a_{ij}]_{m\times n}$ or $(a_{ij})_{m\times n}$

Ex

$$\begin{bmatrix} 1 & 5 & 7 \\ 2 & 8 & 9 \\ 4 & 2 & 7 \end{bmatrix}_{3\times 3} \qquad \begin{pmatrix} 5 & 7 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$

Different types of matrices :-

Row Matrix :-

A matrix having a single row is called row matrix.

→ It is in the form $[a_{ij}]_{1\times n}$ $(a_{11}\ a_{12}\ a_{13})$

Ex $[1 2 3]_{1\times 3}$

Column Matrix :-

A matrix having a single column is called column matrix,

In the form $[a_{ij}]_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

Chapter -1 Matrices

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- A rectangular arrange of elements of the ~~fact~~ form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{m} & a_{m_2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

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Column Matrix :-

A matrix having a single column is called column matrix

In the form $[a_{ij}]_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m\times 1} \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

Rectangulare Matrix :-

A matrix of ordere $m \times n$ is said to be a rectangulare - matrix if $m \neq n$.

Squar Matrix :-

Note :-

The element of Squar matrix $A = [a_{ij}]_{m \times n}$ are classified into 3 types.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{n \times 1}$$

(1) The element $a_{11}, a_{22}, a_{33} \cdots\cdots a_{nn}$ are called digonal element.

(2) The elements a_{ij} $i < j$ are upper digonal elements.

(3) The elements a_{ij}, $i > j$ are lower digonal element.

Digonal Matrix :-

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Scalar Matrix :-

Is matrix said to be a scalar matrix if all the digonal elements as same

Ex

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Null Matrix :-

A matrix said to be a null matrix if all the entrys are zero, is zero Matrix denoted by zero.

Unit Matrix and identy Matrix :-

Is matrix said to be a unit matrix if all the entrys in the liding digonals if $a_{ij} = \begin{cases} 0, & i \neq 0 \\ 1, & j = 1 \end{cases}$

Ex

$$I_1 = [1]_{1\times1}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

Addition of matrix :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Ex

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 5 & 4 & 7 \end{bmatrix} \rightarrow \text{lower}$$

Lower Trangular Matrix :-

- Lower trangular matrix can have non-zero entries only on and below the main digonal.
- Any entries on the main digonal on the trangular matrix may be zero or not other wise a squar matrix if $[a_{ij}] = 0$, $i<j$, that is element above the leading digonal

or if all its upper digonal is zero.

Upper Triangular Matrix :-

Upper triangular matrix are square matrices that can have non - zero matrices. Only on above the main digonal. where, as any entry below the digonal must be zero otherwise a square matrix $[a_{ij}]$ is called an upper triangular matrix. if $a_{ij} =$

that is element below the leading digonal are zero or if all its low digonal element zero.

Ex

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Addition of Matrix :-

Let $A = [a_{ij}]$ be a matrix of order $(m \times n)$ and $B = [b_{ij}]_{m \times n}$ then there addition before

than $A+B = [a_{ij} + b_{ij}]_{m \times n}$ where $i = 1, 2 ------ m$
$j = 1, 2, 3 ---- n$

That is addition of two matrices of same Order is often by adding the element in the corresponding position.

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 7 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 6 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}_{2 \times 3}$$

$$A+B = \begin{bmatrix} 1+6 & 2+1 & 3+2 \\ 5+2 & 3+2 & 7+4 \end{bmatrix}_{2 \times 3}$$

$$= \begin{bmatrix} 7 & 3 & 5 \\ 7 & 5 & 11 \end{bmatrix}_{2 \times 3}$$

Substraction OF Matrix :-

The substraction of two matrix A & B of same order is defend as,

$$\boxed{[A-B] = (A+(-B))}$$

where $(-B)$ is the negative (-ve) matrix of the $[B]$

Transpose Matrixces :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times 2}$$

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 9 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 7 & 9 \end{bmatrix}_{3\times 2}$$

Symetric Matrix :-

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}_{2\times 2} \qquad A^T = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}_{2\times 2}$$

or

$$A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ -3 & -2 & 0 \end{bmatrix}_{3\times 3} \qquad -A = \begin{bmatrix} 0 & -1 & +3 \\ +1 & 0 & +2 \\ \cancel{0}-3 & +2 & 0 \end{bmatrix}_{3\times 3}$$

$$A^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \\ -3 & -2 & 0 \end{bmatrix}_{3\times 3}$$

Matrix multiplication :-

Let $A = [a_{ij}]$ be a matrix of order $m \times n$ and $B = [b_{jk}]$ matrix of order $m \times p$ so that, number of columns in

a 0 ≠

A = number of rows in B. Then the product A, B is well defined and it will be matrix of order $m \times p$, whose - element are given by $c_{ik} = \sum_{j=1}^{n} a \cdot \sum_{j=1}^{n} a_{ij} b_{jk} = a_{ij}$ $b_{ik} + a_{i2} b_{2k} \ldots\ldots + a_{ik} b_{nk}$ (i K) element of A.

Ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times3}$$

AB = $a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{21}$ $a_{11}b_{12} + a_{12}b_{22} + a_{12}b_{22}$

$a_{11}b_{13} + a_{12}b$

AB = $a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{21}$

Properties of matrix multiplication :-

For a any 3 matrix A, B, C conformable for multiplication and Scalar.

① $K(AB) = (KA)B = A(KB)$

② $A(BC) = (AB)C$ (Associative)

③ $(A+B)C = AC+BC$

$C(A+B) = CA + CB$

Sub matrix & minors :-

* Any matrix often by omitting some rules or columns or both of a given m×n matrix 'A' is called a sub-matrix of 'A'. Thus $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is a sub-matrix of
$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$

Rank of a matrix :-

A number 'r' is said to be a ~~matrix~~ rank of a non-zero n×m matrix.

if (i) their is atlest one (r×r) some matrix of a whose determinor is not equal to zero.

* (ii) The determinant of every (r+1) rowed square sub-matrix in a is zero.

How do find rank of matrix :-

To find the rank of matrix, we will transform that matrix into its echelon form in linear algebra matrix a need - echelon from if it has the say resulting from a Gaussian elemination all rows consting of only zero are at the bottom. The leading coefficient (also called pivot of non zero - rows is all ways strictly to the right of the leading coffiecent of the row above it.) then determine the rank of number non-zero rows.

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \end{pmatrix}_{2\times 2}$$

condition :-

* If a matrix if a order $m \times n$ row (c) $P(A) \leq \min(m,n)$ = minimum of (m,n).
* If A is order $m \times n$ an determinant $|A| \neq 0$ then the rank of $A = n$
* If A is of order $m \times n$ & $|A| = 0$, then the rank of A will be less than N.

Ex

find the Rank of

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{3\times 3}$$

Ans

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}_{3\times 3}$$

where $f(A) \leq \min(3,3)$

$= 1(15-16) - 2(10-12) + 3(8-9)$

$= (15-16) - 2(-2) + 3(-1)$

$= -1 + 4 - 3$

$= -4 + 4$

$= 0$

Here A is a singular square matrix in which their is atleast one (2×2) sub matrix.

For example

$$A_1 = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}_{2\times 2}$$

$$15 - 16 = -1 \neq 0$$

Hence, The rank of A is 2. which is less than the order of 3 of singular a square matrix.

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Linear system of eq^{n}s :- (Existance , uniqueness)

A linear eq^n with 'n' unknowns $x_1, x_2 \cdots\cdots x_n$ is an eq^n is of the form $a_1x_1 + a_2x_2 + \cdots\cdots + a_nx_n = b$ —— ①

In the above eq^n if $b=0$ then it is called homogenous linear eq^n.

In contrast eq^n ① is a non homogenous linear eq^n.

Consider a set of 'M' non homogenous eq^n with 'n'.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots\cdots\cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \cdots\cdots\cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 \cdots\cdots\cdots\cdots\cdots\cdots a_{mn}x_n = b_m \end{array}\right\} —— ②$$

* At least one set of values of $x_1, x_2, x_3 \cdots\cdots x_n$ can be found satisfying all the eq^{n}s then the set of eq^{n}s are known as consistent.

If no such set exist then the eq^{n}s are said to be in consistent.

Ex → ① $2x + 5y = 9$
$x - y = 1$

(II) $x + 2y = 7$
$4x + 8y = 28$

(III) $2x + 3y = 5$
$4x + 6y = -8$

① $2x + 5y = 9$

$2 \times (x - y = 1)$

$$\begin{array}{r} \cancel{2x} + 5y = 9 \\ \cancel{2x} - 2y = 2 \\ (-) \quad (+) \quad (-) \\ \hline 7y = 7 \\ \Rightarrow y = 1 \end{array}$$

(Consistent)

② $x + 2y = 7$
$4x + 8y = 28$
$\Rightarrow x = 7 - 2y$
$\Rightarrow$ no. of solⁿs
$\Rightarrow$ consistent

③ $2\times(2x + 3y = 5)$
$4x + 6y = -8$

$\cancel{4x} + \cancel{6y} = 10$
$\cancel{4x} + \cancel{6y} = -8$
$(-) \quad (-) \quad (+)$

$0 = 18$

$\Rightarrow$ This eqⁿ have no solⁿ

$\Rightarrow$ Inconsistent.

The association of homogenous is given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots a_{mn}x_n = 0 \end{array}\right\} \text{———} ③$$

The empresen matrix a given by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} = A$$

is called co-efficent matrix denoted by A.

Then $$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} & b_2 \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix which is denoted by A_b or (A, b)

The system of non-homogenous eqn's ② may be put in the form

$$AX = B$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

consistency of system of linear eqn :-

consider the following m eqn n unknown

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots a_{mn}x_n &= b_m \end{aligned}\right\} \text{———} ①$$

which is in the matrix form

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Here, A is called co-efficient matrix

B is called right an side matrix

with the help of A and B consider

$$K = [A/B]$$

$$\Rightarrow A = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

which is known as augemented matrix.

A sol^n of eq^n (1) defined by set of values of the variable $x_1, x_2, \ldots, x_n$ which satisfy.

If the system given by eq^n (1) has a sol^n, it is called consistent system. Otherwise this is called ~~A~~ inconsistent system.

Infact a consistent system has either unique sol^n or ~~Imphan~~ infinetly many sol^n.

Rouche's theorem :-

The system of equation one if was only co-efficient matrix A and the augmented matrix K are of some rank otherwise the system is in consistent.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$K = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

* Rank of A = Rank of K

$= r\,(r \leq$ the smaller of m & $n)$

The equation ① can by suitable row operations be reduce to

$$b_{11}x_1 + b_{12}x_2 + \cancel{b_{13}} \cdots\cdots\cdots + b_{1n}x_n = k_1$$

And the remaining $(m-r)$ equation being all of the form $\boxed{0x_1 + 0x_2 + \cdots\cdots\cdots + 0x\,r_1 = 0}$

* The equation (2) will have a solution through $n-r$ of the unknown may be choosen arbitrarily. The solution will be unique when $r = n$.
Hence the equations (i) are consistent.

* Rank of 'A' < rank of K. In particular, let the rank of K be $r+1$. In this case the equations (i) will reduce, by suitable row operations to.

$$b_{11}x_1 + b_{12}x_2 + \cdots\cdots + b_{1n}x_n = k_1$$
$$0x_1 + b_{22}x_2 + \cdots\cdots + b_{2n}x_n = k_2$$
$$\vdots$$
$$0x_1 + 0x_2 + \cdots\cdots + b_{rn}x_n = k_r$$
$$0x_1 + 0x_2 + \cdots\cdots + 0x_n = k_{r+1}$$

and remaining $m-(r+1)$ equations are of the form. $0x_1 + 0x_2 + \cdots\cdots + 0x_n = 0$. Clearly the $(r+1)$ the equation can not be satisfied by any set of value for the unknowns. Hence the equations (i) are inconsistent.

__Ex__ (1) Solve
$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$

__Ans__

Given eq^ns are

$$\left.\begin{aligned} x + 2y - z &= 3 \\ 3x - y + 2z &= 1 \\ 2x - 2y + 3z &= 2 \end{aligned}\right\} \text{——— (1)}$$

which can write in the form

$AX = B$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

The given eq^n which can be represented in augmented matrix form

$$K = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$

$\boxed{\sim = \text{equivalent symbol}}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 2 & -2 & 3 & 2 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \end{bmatrix}$$

$(C_2 \leftrightarrow C_3)$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & -8 \\ 0 & 5 & -6 & -4 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & -8 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\rho(K) = 3$

$\rho(A) = 3$

Here $\rho(K) = \rho(A) \Rightarrow$ system are consistent

Next to find the soln of eqn ①

$$x - y + 2z = 3$$
$$5y - 7z = -8$$
$$z = 4$$

Then to find y we can solve.

$$5y - 7z = -8$$
$$\Rightarrow 5y - 7 \times 4 = -8$$
$$\Rightarrow 5y - 28 = -8$$
$$\Rightarrow 5y = 28 - 8$$
$$\Rightarrow 5y = 20$$
$$\Rightarrow y = 20/5 = 4$$

Then,

$$x - y + 2z = 3$$
$$\Rightarrow x - 4 + 2\times4 = 3$$
$$\Rightarrow x - 4 + 8 = 3$$
$$\Rightarrow x + 4 = 3$$
$$\Rightarrow x = -1$$

Hence, $x = -1$
$y = 4$
$z = 4$ are the required solution.

Q. Solve the following system completely.

$$2x - y + 3z = 3$$
$$x + 2y - z - 5w = 4$$
$$x + 3y - 2z - 7w = 5$$

<u>Soln</u> writing the above eqn in matrix form $AX = B$ we get

$$\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 2 & -1 & -5 \\ 1 & 3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

where,

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 2 & -1 & -5 \\ 1 & 3 & -2 & -7 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Here the augmented matrix

$$K = \left[\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3 \\ 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\sim = \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 2 & -1 & 3 & 0 & 3 \\ 1 & 3 & -2 & -7 & 5 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$ & $R_3 = R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & -5 & 5 & 10 & -5 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix}$$

$R_2 = \frac{1}{5} R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & -1 & 1 & 2 & -1 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\rho(K) = 2$

$\rho(A) = 2$

Hence

$\rho(K) = \rho(A)$

So, the equation are

$x + 2y - z - 5w = 4$ —— (1)

$-y + z + 2w = -1$

$\Rightarrow y - z - 2w = 1$ —— (2)

Let $z = k_1$ & $w = k_2$

So the eq^n (2)

$y = 1 + z + 2w$

in eq^n (1) $= 1 + k_1 + 2k_2$

$x + 2(1 + k_1 + 2k_2) - (k_1 - 5k_2) = 4$

$\Rightarrow x + 2 + 2k_1 + 4k_2 - k_1 - 5k_2 - 4$

$\Rightarrow x + k_1 - k_2 - 2$

$\Rightarrow x = 2 + k_2 - k_1$

Hence,

$x = 2 + k_2 - k_1$

$y = 1 + z + 2w$

$z = k_1$ & $w = k_2$

where k_1 and k_2 is constant.

chapter - 2 COMPLEX NUMBER

Real Number :-

Real numbers are numbers that includes both rational and irrational numbers. Rational numbers such as integer ($-2, -1, 0, 1, 2$ etc), fractions ($1/2$, $5/7$, 2.5, 7.1) and irrational numbers such as ($\sqrt{3}$, $\sqrt{5}$, $\sqrt{2}$, $\pi\ (\frac{22}{7})$).

* Square of a positive real number is positive and that of a negative real is also positive. So, there is no real number whose square is negative. So, we are two create a new kind of number. We define a square root of a negative number as imaginary number particularly $\sqrt{-1} = i$ the basic imaginary number. Then, $\sqrt{-4} = 2i$, $\sqrt{-9} = 3i$. $\sqrt{-2} = \sqrt{2}\,i$ etc.

Taking $i = \sqrt{-1}$

$$i^2 = \sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1$$

$$i^3 = i^2 \times i = -i$$

$$i^4 = i^2 \times i^2 = 1$$

Since $i^4 = 1$,

$$i = i^5 = i^9 = i^{13} + \ldots\ldots\ldots i^{4n+1}$$

when n is integer.

$$i^2 = i^6 = i^{10} = \ldots\ldots\ldots = i^{4n+2}$$

$$i^3 = i^7 = i^{11} = \ldots\ldots\ldots = i^{4n+3}$$

$$i^4 = i^8 = i^{12} = i^{16} = \ldots\ldots\ldots = i^{4n}$$

Complex Number :-

The number of the form $a+ib$.

where a & b are real numbers & $i = \sqrt{-1}$ are known as complex number.

$$\underbrace{(a)}_{\text{real part}} + i\,\underbrace{(b)}_{\text{imaginary part}}$$

In complex number $z = a + ib$, the real numbers a & b are respectively known as real and imaginary part of z and we write.

$$Re(z) = a$$
$$\& \ Im(z) = b$$

Thus, the set C of all complex number is given by

$$C = \{z : z = a + ib \ ; \text{ where } a, b \in R\}$$

Purely Real and Purely Imaginary numbers :-

A complex number z is said to be.

① Purely real if $im(z) = 0$ (ex $-2, -3, \sqrt{3}$ etc)

② Purely imaginary if $re(z) = 0$ (ex $-2i, -7i, \sqrt{3}i, 1/7 i$ etc)

Conjugate of a complex number :-

The conjugate of a complex number z denoted by $\bar{z}$ is the complex number obtained by changing the sign of imaginary part of z.

Ex -

① $z = 2 + 5i$
$\bar{z} = \overline{(2+5i)}$
$= 2 - 5i$

② $z = 7i$
$\bar{z} = \overline{(7i)}$
$= -7i$

③ $z = \sqrt{3} + 7i$
$\bar{z} = \overline{(\sqrt{3} + 7i)}$
$= \sqrt{3} - 7i$

④ $z = -9i$
$\bar{z} = \overline{(-9i)}$
$= 9i$

Modulus of a complex number :-

If $z = x + iy$ be a complex number, the modulus of z written as $|z|$ is a real number $\sqrt{x^2 + y^2}$

Ex - ① $z = 3 + 4i$
$|z| = \sqrt{3^2 + 4^2}$
$= 5$

② $z = 6 + 2i$
$|z| = \sqrt{6^2 + 2^2}$
$= 2\sqrt{10}$

* Also $|z| = |\bar{z}|$

Equality of complex number :-

Two complex numbers $z_1 = a_1 + ib_1$

& $z_2 = a_2 + ib_2$ are said to be equal if their real part most be equal to it's real part and imaginary part is most be equal to it's imaginary part that is $a_1 = a_2$ and $b_1 = b_2$.

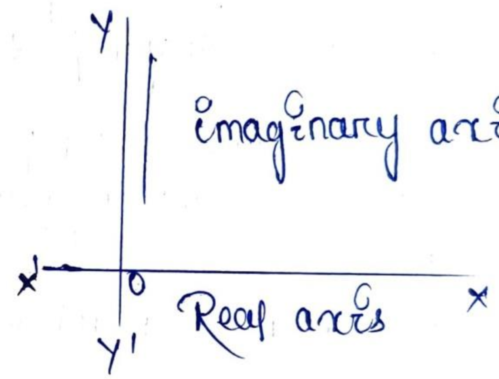
That is

$$Re(z_1) = Re(z_2)$$

$$\& \ Im(z_1) = Im(z_2)$$

Geometrical representation of complex number :-

Complex numbers as order pairs :-



We know that a complex number is of the form $z = a + ib$ where $a \& b$ are real numbers thus, corresponding to each $z = a + ib$ there is associated a unique order pair (a, b) of real numbers. So, we may represent

$$z = a + ib \text{ by } (a, b)$$

Thus, if $z_1 = a, b$ and $z_2 = c, d$ then we may define

$$z_1 + z_2 = (a, b) + (c, d)$$

$$(a + c, b + d)$$

$$z_1 z_2 = (a + ib)(c + id)$$

$$= ac + iad + ibc + i^2 bd$$

$$= ac + (-1) bd + ibc + ibd$$

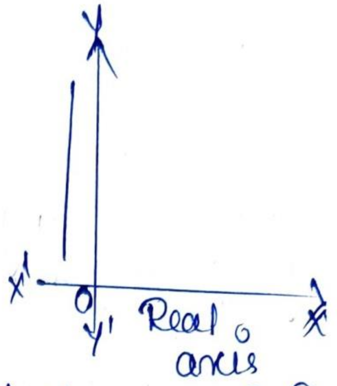
$$= ac - bd + i(ad + bc)$$

clearly $z_1 \Leftrightarrow z_2$

$\Leftrightarrow (a, b) = (c, d) \quad \Leftrightarrow a = c, b = d$

Geometrical Representation :-

Let 'O' be the origine 'x'Ox' & 'y'Oy' be the co-ordinate axis. The real numbers are taken along x-axis and the imaginary numbers ~~are taken along~~ y axis. So, the x axis is called the real axis and the y axis is called the imaginary axis. Then, any complex number $z = a + ib$ may be represented by a unique point $P(a,b)$ whose co-ordinates are (a,b) order pair. The ~~rep~~ -representation of a complex number as points in a plane forms and Argand diagram.



* The plane on which complex number as represented is known as the complex plane or Argands plane or Gaussian plane.

* Let 'a' be a real number then we can write -

$a = a + i0$
$= (a, 0)$

Let $x'Ox$ and $y'Oy$ be the co-ordinate axis.
Let $z = a + ib$ be the complex number represented by point $P = (a, b)$
Draw $PM \perp OX$
then,
$OM = a$ and $PM = b$ join OP
Let $OP = r$ and $\angle XOP = \theta$ then $a = r\cos\theta$ and $b = r\sin\theta$.

$$z = a + ib$$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{a^2+b^2} = |z|$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{r}}{\frac{a}{r}} = \frac{b}{a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

The form $z = r(\cos\theta + i\sin\theta)$ is called the polar form or Trigonometrical or standard form or modulus form or Amplitude form of z.
Here, $r = |z|$ and the angle θ is known as amplitude or argument of z.
written as amp (z) or arg (z).

The unique value of θ such that $-\pi < \theta \leq \pi$ for which $a = r\cos\theta$ or $b = r\sin\theta$ is known as the principle-value of amplitude.

The general value of amplitude is $(2n\pi + \theta)$ where n is an integer and θ is a principle value of.

Theorem :- 1

If $z \in \mathbb{C}$ then

① $\overline{(\bar{z})} = z$

② $(z + \bar{z}) = 2\,Re(z)$
which is real

③ $(z - \bar{z}) = 2i\,Im(z)$
which is an imaginary no.

④ $(z\bar{z}) = |z|^2$ & therefor
$z\bar{z}$ is real

⑤ $Re(z) \leq |z|$ & $Im(z) \leq |z|$

Let $z = a + ib$
$\bar{z} = a - ib$

① $\overline{(\bar{z})}$
$= \overline{(a - ib)}$
$= a + ib = z$

② $(z + \bar{z})$
$= (a + ib) + (a - ib)$
$= a + \cancel{ib} + a - \cancel{ib}$
$= 2a = 2Re(z)$

③ $(z - \bar{z})$
$= (a + ib) - (a - ib)$
$= \cancel{a} + ib - \cancel{a} + ib$
$= 2ib$

④ $(z\bar{z})$
$= (a + ib)(a - ib)$
$= (a)^2 - (ib)^2$
$= a^2 - i^2 b^2$
$= a^2 - (-1)^2 b^2$
$= a^2 + b^2 = |z|^2$

⑤ Let $z = x + iy$
$|z| = \sqrt{x^2 + y^2}$
$x^2 \leq x^2 + y^2$
$\Rightarrow x \leq \sqrt{x^2 + y^2}$
i.e $Re(z) \leq |z|$
$y^2 \leq x^2 + y^2$
$\Rightarrow y \leq \sqrt{x^2 + y^2}$ i.e $Im(z) \leq |z|$

Theorem:-2

If z_1 & $z_2 \in C$ then,

① $\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$

② $\overline{(z_1-z_2)} = \bar{z}_1 - \bar{z}_2$

③ $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$

④ $|z_1 z_2| = |z_1||z_2|$

Proof

Let $z_1 = a+ib$

& $z_2 = c+id$

① $z_1+z_2 = (a+c) + i(b+d)$

$\overline{(z_1+z_2)} = \cancel{a+c-ib-id}\ a+c-i(b+d)$

$= a+c-ib-id$

$= (a-ib)+(c-id)$

$= \bar{z}_1 + \bar{z}_2$ [Proved]

② $z_1-z_2 = (a-c)+i(b-d)$

$\overline{(z_1-z_2)} = a-c+ib-id$

$= (a+ib)-(c-id)$

$= \bar{z}_1 - \bar{z}_2$ [Proved]

③ $\overline{(z_1 z_2)} = \bar{z}_1 \cdot \bar{z}_2$

Let $z_1 = (a+ib)$ & $z_2 = (c+id)$

$z_1 z_2 = (a+ib)(c+id)$

$= (ac-bd)+i(ad+bc)$

$= (a-ib).(c-id)$

$\overline{z_1 z_2} = (ac-bd) - i(ad+bc)$

$= (a-ib).(c-id)$

$= \overline{(a+ib)} \cdot \overline{(c+id)}$

$= \bar{z}_1 . \bar{z}_2$ [Proved]

④ $|z_1 z_2| = |z_1| . |z_2|$

Let $z_1 = (a+ib)$ & $z_2 = (c+id)$

$|z_1| = \sqrt{a^2+b^2}$

$|z_2| = \sqrt{c^2+d^2}$

$z_1 z_2 = (a+ib)(c+id)$

$= (ac-bd)+i(ad+bc)$

$|z_1 z_2| = \sqrt{\{(ac-bd)^2+(ad+bc)^2\}}$

$= \sqrt{a^2+b^2} . \sqrt{c^2+d^2}$

$= |z_1| . |z_2|$ [Proved]

Theorem :- 3

$$\boxed{z\bar{z} = |z|^2}$$

$|z_1 + z_2| \le |z_1| + |z_2|$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_2|^2$

$|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + (z_1\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2) \le |z_1|^2 + |z_2|^2 + 2|z_1|^2 \cdot [\because Re(z) \le |z|]$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cdot [\because |z_2| = |\bar{z}_2|]$

$= (|z_1| + |z_2|)^2$

Reciprocal of a complex number :-

Let $z = a + ib$ then the reciprocal of $z = 1/z$

$$= \frac{1}{a+ib}$$

$$= \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$$

$$= \frac{a-ib}{a^2 - i^2b^2}$$

$$= \frac{a-ib}{a^2-b^2}$$

$$= \left(\frac{a}{a+b^2}\right) - i\left(\frac{b}{a^2+b^2}\right)$$

Ex-1

$$z = a + ib$$

$$z = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow 1 + i = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow 1 + i = r\cos\theta + ir\sin\theta$$

$$r\cos\theta = 1$$

$$r\sin\theta = 1$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2$$

$$\Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\cos\theta = 1/\sqrt{2}$$

$$\sin\theta = 1/\sqrt{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$1 + i = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Hence the polar form of $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Chapter - 3 Differential Equations

Solutions of a Differential Equations :-

The solution of a general ordinary differential equation of nth order.

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\ldots \frac{d^ny}{dx^n}\right) = 0$$

$$\text{or } F(x, y, y', y'' \ldots\ldots y^{(n)}) = 0 \ldots\ldots (1)$$

Defination :-

Let $y = f(x)$, define y as a real function of x on a real interval I. Then f in called an explicit soln or simply a solution of the differential equation (1), and if we put $y = f(x)$ and the given equation which is of the form,

$f((x, f(x), f'(x) \ldots\ldots f^n(x)) = 0$

Defination :-

A relation $g(x, y) = 0$ is called implicit statisfied sol^n of the differential equation by putting $y = \phi(x)$ and 'I' such that ϕ is an explicit soln.

Any relation between the dependent variables not involving the dereivatives which, when sabstituted in the differential eqn reduces it to an identity is called solution of the differential equation.

General Solution : (or complete primitive) :-

The general solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation and which satisfies the given differential equation.

PRIMITIVE OR SOLUTION OF A DIFFERENTIAL EQUATION

A primitive or solution of a differential equation is a functional such that this relation and the derivatives obtained ~~form~~ from it satisfy the given differential equation.

For example, $x = \cot y + c$ is the solution of the differential equation $\frac{dy}{dx} + \sin^2 y = 0$

Now $x = \cot y + c$ gives us

$$\frac{dy}{dx} = -\operatorname{cosec}^2 y$$

or $$\frac{dy}{dx} = -\sin^2 y$$

Substituting the value of $\frac{dy}{dx}$ in L.H.S. of differential equation, we get

$$-\sin^2 y + \sin^2 y = 0$$

$$L.H.S = R.H.S$$

Thus the sol^n of a differential eq^n is a functional relation between x and y which is free from derivatives and this relation on substitution satisfy the differential eq^n.

General sol^n :-

It is that sol^n which contains the number of arbitrary constants equal to the order of the differential eq^n. It is also called complete primitive.

Thus is the above example the sol^n contains on arbitrary constant and the eq^n is of first order.

Particular sol^n :-

A particular sol^n of differential eq^n is sol^n obtained from the general sol^n by giving particular values to the arbitrary constant. For example, putting $c = 1, 2$ etc, we have $x = \cot y + 1$, $x = \cot y + 2$ which are particular

Sol^n of the eq^n $\frac{dy}{dx} + \sin^2 y = 0$

Note :- In exceptional cases a relation containing 'n' arbitrary constants may give rise to differential eq^n of order less than n.

Formation of a differential eq^n :-

we have seen that a general sol^n to a differential eq^n is a relation bet^n the variables and it contains the number of arbitrary constants equal to the - order of the eq^n. As we have to study the - differential eq^n of the first order, let us have a functional relation

$$F(x, y, c) = 0 \quad \text{———} \quad (1)$$

Now we shall form that differential eq^n whose sol^n is

$$F(x, y, c) = 0$$

The required differential eq^n will be obtained by eliminating c from eq^n (1) and another eq^n obtained by differentiation (1) w.r.t. x.

In order words we have to eliminate c from the eq^{n}s.

$$F(x, y, c) = 0$$

Eliminating of 'c' from those two equations, we get the required differential eq^n.

Date - 29.11.2022

Example-1

Q. Find the differential eq^n of the family of curves
$y = e^x (A\cos x + B\sin x)$

Sol^n: given that the family of curves $y = e^x(A\cos x + B\sin x)$
differentiate w.r.t x.

$$\frac{dy}{dx} = \frac{d(e^x)}{dx}(A\cos x + B\sin x) + e^x\left\{\frac{d(A\cos x)}{dx} + \frac{d(B\sin x)}{dx}\right\}$$

$$= e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$\Rightarrow \frac{dy}{dx} = y + e^x(-A\sin x + B\cos x)$$

$$\Rightarrow \frac{dy}{dx} - y = e^x(-A\sin x + B\cos x)$$

again differentiate w.r.t x

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x\{-A\cos x - B\sin x\}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-A\cos x + B\sin x) - y$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = \frac{dy}{dx} - y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the require ordinary differential eq^n.

Soln of the differential eqn of First Order and First degree :-

Type 1 :

Equation of the type $\frac{dy}{dx} = F(x)$

$$\frac{dy}{dx} = F(x)$$

$$\Rightarrow dy = F(x).dx$$

integrating both side we get.

$$\Rightarrow \int dy = \int F(x) dx$$

$$\Rightarrow y = \phi(x) + c \text{ where } \phi(x) = \int F(x) dx$$

Type 2 :

Equation of the type $\frac{dy}{dx} = F(y)$

$$\frac{dy}{dx} = F(y)$$

$$\Rightarrow \frac{dy}{F(y)} = dx$$

$\Rightarrow$ on integrating

$$\Rightarrow \ln(f(y)) + c = x$$

$$\Rightarrow F(y) = e^{x} + c$$

Type 3 :

Equation with variables separable :-

If a given differential equation of a being expressed in the form $F(x).dy + g(y).dx = 0$

$$\text{i.e } f(x) dy + g(y).dx = 0$$

$$f(x) dy = -g(y).dx$$

$$\Rightarrow \frac{dy}{g(y)} = \frac{-dx}{f(x)}$$

On intigrating

$\Rightarrow \ln(g(y)) = -\ln(F(x)) + \ln c$

$\Rightarrow \ln(g(y)) + \ln(F(x)) = \ln c$

$\Rightarrow \ln(g(y)\,F(x)) = \ln c$

where $g(y)$ & $F(x)$ are respectively functions of y & x is called a variable and separatable

The solⁿ of such an eqⁿ is obtain by intigrating each term separately.

Type - 4 :

Solving of differential eqⁿ of second order of

$$\frac{d^2y}{dx^2} = f(x)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x)$$

On intigration, both side

$$\Rightarrow \frac{dy}{dx} = \int f(x) \cdot dx + c_1$$

let $\int f(x) \cdot dx = \phi x$

$$\Rightarrow \frac{dy}{dx} = \phi(x) + c_1$$

now multiplying both side with differential dx and the the intigrating both side.

$$\Rightarrow \int dy = \int (\phi(x) + c_1) \cdot dx$$

$$\Rightarrow y = \int \phi(x)\, dx + c_2 + c_1 x$$

$$\Rightarrow y = \int \phi(x)\, dx + c_1 x + c_2$$

$$= \psi(x) + c_1 x + c_2$$

where $\psi(x) = \int \phi(x)\, dx$.
c_2 is the constant integration which is independed c_1.

$y = \psi x + c_1 x + c_2$ is the require differential eqⁿ.

Type-5 :-

Equation reducible to variable separable
The eqⁿ of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduce to variable separable by the substitution
$ax + by + c = z$

Type 6 :-

Homogenous function.
A function $f(x,y)$ in x and y is called a homogenous function of degree n, if the degree of each term is n.

Example -1 :-
$f(x,y) = x^2 + y^2 - xy$ is a homogenous function of degree 2.

Example - 2 :-
$f(x,y) = x^3 + 5x^2y + x^2y$ is a homogenous funⁿ of degree 3.

Degree of Homogeneous differential eqⁿ :-
A differential eqⁿ of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where $f(x,y)$ as well as $g(x,y)$ is a homogeneous funⁿ of same degree in x and y is called a homogeneous differential eqⁿ.

Ex

① $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

② $\frac{dy}{dx} = x \cos(y/x)$ is a homogenous differential eqn as $x \cos(y/x)$ being a funn of y/x is a homogenous funn.

Method of solving a homogenous eqn :-

Let $\frac{dy}{dx} = f(x,y)\ g(x,y)$ is a homogenous differential eqn.

Put $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Subtitute this values in the given eqn we get

$v + x\frac{dv}{dx} = f(v)$

$\Rightarrow x\frac{dv}{dx} = f(v) - v$

$\Rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$

On integration both side

$\Rightarrow \int \frac{dv}{f(v)-v} = \int \frac{dx}{x}$

$\Rightarrow \ln + c$

Now replace v by y/x to obtain the requiere soln.

Type 7 :

Equation reducible to homogeneous function.
Equation of the type.

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + c}$$

$x = X + h$

$y = Y + K$

where h & K are constant.

$$\frac{dy}{dx} = \frac{d(Y+K)}{dx} = \frac{d(Y+K)}{dX}\cancel{\frac{dX}{dx}}^{1} = \frac{dY}{dX}$$

$$\frac{ax + by + c}{Ax + By + c}$$

$$= \frac{a(X+h) + b(Y+K) + c}{A(X+h) + B(Y+K) + c}$$

$$= \frac{aX + bY + (ah + bK + c)}{AX + BY + (Ah + BK + c)}$$

$$\begin{matrix} A \\ a \end{matrix}\left(\begin{matrix} ah + bK + c = 0 \\ Ah + BK + c = 0 \end{matrix}\right)$$

$\Rightarrow \cancel{Aah} + AbK + cA = 0$

$\quad \underset{(-)}{\cancel{aAh}} \underset{(-)}{+ aBK} \underset{(-)}{+ ac} = 0$

$$AbK - aBK = ac - cA$$

$$\Rightarrow K = \frac{ac - cA}{Ab - aB}$$

Exact Equation :-

The differencial eq^n $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if and only $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where, $\frac{\partial M}{\partial y}$ denotes the differential co-efficent of ~~M~~ M with respect to y keeping x constant.

Rule-1

Rule-2

~~Intig~~ Integrate w.r.t y. only those terms of N which do not contain x.

Rule-3

Result of ① + Result of ② = constant

Integrating factor :-

An integrating factor is a function when multiplied by it the left hand side of eq^n $M(x,y)\,dx + N(x-y)\,dy = 0$ becomes the exact equation.

$M(x,y)\,dx + N(x-y)\,dy = 0$ — ① is not exact or total differential. It is easy to choose func^n $\mu(x,y)$ such the after multiplying by it the left side of eq^n ① becomes the exact differential $du = \mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy$

Note:-

① The number of integrating factor is infinite.

② If $Mx + Ny \neq 0$ and eq^n ① is homogenous then $\frac{1}{Mx+Ny}$ is the integrating factor of the ~~integrating~~ integration.

③ $Mx + Ny = 0$

$\Rightarrow Mx = -Ny$

$\Rightarrow \frac{M}{N} = \frac{-y}{x}$

④ If $Mx - Ny \neq 0$ and the eqn — ① has the form $M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{1}{Mx - Ny}$ is the intigrating factor.

Linear differential eqn :-

A differential eqn in which the dependent variables and the both occur in the first degree only and are not multiplied together, is called a linear differential eqn.

$\frac{dy}{dx} + 5y = x^2$ is a linear eqn of order 1.

$x\frac{d^2x}{dy^2} + 5\frac{dy}{dx} = 8$ is a linear eqn of order degree 2.

$x\left(\frac{dy}{dx}\right)^2 - y^2\frac{dy}{dx} + 8 = 0$ is a Non-linear differential eqn of order 2 degree 2.

Note :-

Every linear differential eqn is of degree 1 but every differential eqn of degree 1 is need not linear.

Type 1 :

$\frac{dy}{dx} + Py = Q$, where P is a constant and Q may be a constant or a funn of x only.

To solve $\frac{dy}{dx} + Py = Q$,

First we find $e^{\int P dx}$, which is known as integrating factor, written as I.F.
Multiplying both sides of the given eqⁿ by $e^{\int P dx}$ we get

$$e^{\int P dx} \cdot \frac{dy}{dx} + Py\, e^{\int P dx} = Q\, e^{\int P dx}$$

or, $e^{\int P dx} \frac{dy}{dx} + Py\, e^{\int P dx} = Q e^{\int P dx}\, dx$

or, $\frac{d}{dx}\left(y\, e^{\int P dx}\right) = Q e^{\int P dx}\, dx$

Integrating, we get

$$y \cdot e^{\int P dx} = \int Q e^{\int P dx}\, dx + c$$

which is the required solⁿ of the given diffⁿ eqⁿ.

Working Rule for solving $\frac{dy}{dx} + Py = Q$

(i) Find $I.F = e^{\int P dx}$

(ii) The solⁿ is $y \times (I.F) = \int \{Q \times (I.F)\}\, dx + c$

Type 2:

Differential eqⁿs linear in x and $\frac{dx}{dy}$

These eqⁿs are of the form $\frac{dx}{dy} + Px = Q$

where P and Q are funⁿs of y only or constants

The solⁿ is given by integrating factor $= I.F. = e^{\int P dy}$

Next to find it's solⁿ

$$x\, e^{\int P dy} = \int Q \cdot e^{\int P dy} \cdot dy + c$$

$$x\,(I.F) = \int Q \cdot (I.F)\, dy + c$$

Type 3 :

Equations reducible to the linear Form :

(a) $\frac{dy}{dx} + Py = Qy^n$

where P and Q are constants or fun's of x alone and n is a constant other than zero or unity can be reduced to the linear form by y^n and Substituting $\frac{1}{y^{n-1}} = z$

On dividing the given eqⁿ by y^n, we get

$$\frac{1}{y^n}\left(\frac{dy}{dx}\right) + \frac{1}{y^{n-1}} P = Q$$

Put $\frac{1}{y^{n-1}} = z$, then $\frac{-n+1}{y^n} \frac{dy}{dx} = \frac{dz}{dx}$ and

(2) becomes $\frac{1}{(-n+1)} \frac{dz}{dx} + Pz = Q$, which is linear eqⁿs:

(b) Similarly $f'(y) \frac{dy}{dx} + Pf(y) = Q$, can be reduced to the linear form by the Substitution $f(y) = z$.

Example 1:

Solve $\frac{dy}{dx} + xy = xy^3$

Dividing by y^3, the eqn becomes

$\Rightarrow \frac{dy}{dx} + \frac{3}{x} y = x$

where $p = 3/x$ & $Q = x$

$I.F = e^{\int p.dx} = e^{\int \frac{3}{x}.dx} = e^{3\ln x + c}$

$= e^{\ln x^3} = x^3 + c$

So the required soln is

$I.F\ xy = \int (Q \times I.F)\, dx + c$

$\Rightarrow x^3 xy = \int (x \times x^3)\, dx + c$

$\Rightarrow x^3 y = \frac{x^5}{5} + c$

Example 2:

Solve $\frac{dy}{dx} + (\sec x) y = \tan x$

clearly from the above differential eqn

we get $P = \sec x$ & $Q = \tan x$

Then, $I.F = e^{\int P dx} = e^{\int \sec x . dx} = e^{\ln|\sec x + \tan x|} + c$

$= (\sec x + \tan x) + c$

So, the required soln is

$I.F\ xy = \int Q + I.F\ dx + c$

$y(\sec x + \tan x) = \int \tan x (\sec x + \tan x)\, dx + c$

$y(\sec x + \tan x) = \int \sec x \tan x \, dx + \int \sec^2 x \, dx - \int dx + c$

or $y(\sec x + \tan x) = \sec x + \tan x - x + c.$

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Ex - 1

$(x + 2y^3) \frac{dy}{dx} = y$

$\Rightarrow x + 2y^3 = y \cdot \frac{dx}{dy}$

$\Rightarrow y \cdot \frac{dx}{dy} - x = 2y^3$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$

$P = -1/y$ & $Q = 2y^2$

$I.F = e^{\int P \cdot dy}$

$= e^{\int -1/y \, dy}$

$= e^{-\int 1/y \cdot dy}$

$= e^{-\ln y}$

$= e^{\ln y^{-1}}$

$= y^{-1}$

$= 1/y$

Thus, the sol^n is

$I.F. \times x = \int (Q \times I.F)\, dy + c$

$\Rightarrow \frac{x}{y} = \int \left(2y^2 \times \frac{1}{y} \right) dy + c$

$= 2 \int y \cdot dy + c$

$= y^2 + c$

$\Rightarrow x = y\,(y^2 + c)$

Ex-2

Solve $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$

$\Rightarrow (1+y^2) \frac{dx}{dy} = \tan^{-1} y - x$

$\Rightarrow (1+y^2) \frac{dx}{dy} - x = \tan^{-1} y$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$

$P = \frac{1}{1+y^2} \qquad Q = \frac{\tan^{-1} y}{1+y^2}$

$I.F \cdot e^{\int P.dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Thus the solⁿ is

$I.F \times x = \int (Q \times I.F)\, dy + c$

$x \cdot e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2} \times \cancel{e^{\tan}}\, e^{\tan^{-1} y} \right) dy + c$

$\left[\text{Put } z = \tan^{-1} y \qquad \frac{dy}{1+y^2} = dz \right]$

$\Rightarrow x \cdot e^z = \int z \cdot e^z \cdot dz + c$

$= \int z e^z - e^z + c$

$= x \cdot e^z = e^z (z-1) + c$

① For any $z_1, z_2 \in \mathbb{C}$, prove that

① $|z_1+z_2| \le |z_1|+|z_2|$

② $|z_1-z_2| \le |z_1|+|z_2|$

A. i) $|z_1+z_2|^2 = (z_1+z_2)\,(\overline{z_1+z_2})$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$

$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + (\overline{z_1\bar{z}_2})$

$= |z_1|^2 + |z_2|^2 + 2\,Re\,(z_1\bar{z}_2) \le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$= |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$

$= |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|$

$= (|z_1|+|z_2|)^2$

$\therefore |z_1+z_2| \le |z_1|+|z_2|$

ii) $|z_1-z_2| = |z_1+(-z_2)| \le |z_1|+|-z_2|$

$= |z_1|+|z_2|$

$\therefore |z_1-z_2| \le |z_1|+|z_2|$

② $|z_1 z_2| = |z_1||z_2|$ & $Arg\,|z_1 z_2| \ne arg(z_1) + arg(z_2)$

A. Let
$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_1 + i\sin\theta_2)$

Then,

$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_1 + i\sin\theta_2)$

$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (i\sin\theta_1\cos\theta_2 +$
$\cos\theta_1 + \sin\theta_2)]$

$$= r_1 r_2 \left[\cos(\theta_1+\theta_2) i\sin(\theta_1+\theta_2)\right]$$

$$|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|$$

and $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1)$

(iii) $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \frac{(\cos\theta_1 + i\sin\theta_1)\left[\cos(-\theta_2) + i\sin(-\theta_2)\right]}{\left[\cos^2\theta_2 + \sin^2\theta_2\right]}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

and, $\arg\left|\frac{z_1}{z_2}\right| = (\theta_1 - \theta_2) = \left[\arg(z_1) - \arg(z_2)\right]$

3) If $z_1, z_2 \in C$, then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

A. let,

$z_1 = a + ib$

$z_2 = c + id$

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$= \frac{(a+ib)(c-id)}{c^2+d^2}$$

$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$$

$$\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{bc-ad}{c^2+d^2}\right)^2}$$

$$= \sqrt{\frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}}$$

$$= \sqrt{\left(\frac{a^2+b^2}{c^2+d^2}\right)}$$

$$= \frac{\sqrt{(a^2+b^2)}}{\sqrt{c^2+d^2}}$$

$$= \frac{|z_1|}{|z_2|}$$

Hence $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(5.) $\frac{(x+i)^2}{x-1} - \frac{(x-\sqrt{-1})^2}{x+i}$

$$= \frac{(x+i)^2}{x-i} - \frac{(x-i)^2}{x+1}$$

$$= \frac{(x+i)^3 - x(x-i)^3}{(x-i)(x+i)}$$

$$= \frac{\{(x+i-x+i)(x+i)^2+(x-i)^2+(x+i)(x-i)\}}{x^2-i^2}$$

$$= 2i\frac{\{x^2+i^2+2ix+x^2+i-2xi+x^2-i^2\}}{x^2+1}$$

$$= \frac{2i(3x^2+i^2)}{x^2+1}$$

$$= \frac{6x^2i+2i^3}{x^2+1}$$

$$= \frac{(6x^2-2)i}{x^2+1}$$

(6) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$

A. LHS

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$

$=(1-\omega)(1-\omega^2)(1-\omega^3.\omega)(1-\omega^3.\omega^2)$

$=(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$=(1-\omega^2)(1-\omega^2)^2$

$=\{(1-\omega)(1-\omega^2)\}^2$

$=(1-\omega-\omega^2+\omega^3)^2$

$=(1-\omega-\omega^2+1)^2$

$=(2-\omega-\omega^2)^2$

$=(2+1)^2=3^2=9$ RHS

(10) Find the rank of the given matrix.

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

A. Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \quad [C_3 \to C_3 - C_1 , C_4 \to C_4 - C_1]$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} C_3 \to C_3 + 3C_2 \\ C_4 \to C_4 + C_2 \end{bmatrix}$$

Hence rank of the matrix is 2.

(11) Find the rank of the following matrices $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

A. $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$ $\begin{bmatrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ $\left[R_3 \to R_3 - \frac{1}{2} R_2 \right]$

Hence rank of matrix is 2.

(12) Solve $x + 2y - z = 3$
$3x - y + 2z = 1$
$2x - 2y + 3z = 2$

A. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$K = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \end{bmatrix}$

$R_2 = R_2 - 3R_1$

$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 2 & -2 & 3 & 2 \end{bmatrix}$

$R_3 = R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \end{bmatrix}$$

$(C_2 \leftrightarrow C_3)$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & -8 \\ 0 & 5 & -6 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & 8 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\rho(A) = 3$ Obviously rank of $\rho(K)$ matrix is also 3

$\rho(A) = \rho(K) =$ eq^{n}s are ~~constan~~ consistent.

Next to find its ~~eq~~ sol^n.

$$x - y + 2z = 3, \quad 5y - 7z = 8, \quad z = 4$$

By backward substitution we get $z = 4$

$5y - 7z = -8$, $5y - 28 = -8$, $5y = 20$, $y = 4$

$x - 4 + 8 = 3$, $x + 4 = 3$, $x = -1$

or $x = -1$, $y = 4$, $z = 4$ be the required solutions.

Rules for finding the complementary function :-

To solve the equation

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots K_n y = 0 \ldots\ldots (i)$$

where K_1 s are constants. The eqn (i) in symbolic form is

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \ldots\ldots + K_n) y = 0$$

It's symbolic co-efficients are equated to zero i.e.

$$D^n + K_1 D^{n-1} + K_2 D^{n-2} + \ldots\ldots + K_n = 0 \ldots\ldots (ii)$$

is called auxiliary eqn (A.E) or characteristic eqn.

Let $y = e^{mx}$ be the soln of eqn (ii), then substituting

$y = e^{mx}$, $Dy = me^{mx}$, $D^2y = m^2 e^{mx}$

$D^{n-1} = m^{n-1} e^{mx}$, $D^n y = m^n e^{mx}$ in eqn (ii) we get

$$(m^n + K_1 m^{n-1} + K_2 m^{n-2} + \ldots\ldots + K_{n-1} m + K_n) e^{mx} = 0$$

Since $y = e^{mx}$ is a soln of (ii)

$$m^n + K_1 m^{n-1} + \ldots\ldots + K_n = 0$$

i.e if 'm' is a root of the eqn, then

$$D^n + K_1 D^{n-1} + \ldots\ldots + K_{n-1} D + K_n = 0$$

Let $D = m_1, m_2 \ldots\ldots m_n$ be the roots of the auxiliary eqn.

Case 1: The roots of the A.E are all real and different.

Since the roots $m_1, m_2, \ldots\ldots m_n$ of the A.E are all real and different then eqn (ii) may be written as

$$(D - m_1)(D - m_2) \ldots\ldots (D - m_n) y = 0 \ldots\ldots (iii)$$

Since eqn (iii) will be satisfied by the soln of the eqns.

$$(D - m_1) y = 0, \ (D - m_2) y = 0 \ldots\ldots (D - m_n) y = 0 \ldots\ldots (iii)$$

Let us consider the eqn

$$(D - m_1) y = 0 \Rightarrow \frac{dy}{dx} - m_1 y = 0 \ldots\ldots (iv)$$

This is Leibnitz is linear eqn, I.F $= e^{-\int m_1 dx} = e^{-m_1 x}$

Thus the solⁿ of the eqⁿ (iv) is $y.e^{-m_1x} = \int 0.e^{-m_1x}dx = \int 0\,dx = c_1$

$y = c_1e^{m_1x}$

Similarly considering other factors $y = c_2e^{m_2x}$, $y = c_3e^{m_3x}$ ...... $y = c_ne^{m_nx}$

Hence the complete solⁿ is $y = c_1e^{m_1x} + c_2e^{m_2x} + \dots + c_ne^{m_nx}$ ......(v)

Case 2: The two of the roots of the auxiliary eqⁿ are equal:

Let $m_1 = m_2$

Then the solⁿ obtained in (v) is not a general solⁿ because in this case the result (v) becomes

$$y = (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$$

Thus there are $(n-1)$ arbitrary constants and not 'n'

To find the general solⁿ let us consider $(D - m_1)^2y = 0$

Let $(D - m_1)y = v$

Then the eqⁿ reduces to

$(D - m_1)v = 0$ i.e $\frac{dv}{dx} - m_1v = 0$

whose solⁿ is $v = c_1e^{m_1x}$

Substituting this values $\frac{dy}{dx} - m_1y = c_1e^{m_1x}$ of v as $(D - m_1)y$, we get

The eqⁿ is in Leibnitz's form, $I.F. = e^{-\int m_1dx} = e^{-m_1x}$

Next to find it's solⁿ

$y(I.F) = \int (I.F)\,Q.dx \Rightarrow y.e^{-m_1x} = \int e^{-m_1x}.c_1e^{m_1x}dx$

$y.e^{-m_1x} = c_1\int dx = c_1x + c_2$

Hence it's complete solⁿ is $y = (c_1x + c_2)e^{m_1x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$

However in this case three roots of the A.E are equal.

Say $m_1 = m_2 = m_3$

then proceeding as above the solⁿ becomes

$$y = (c_1x + c_2x^2 + c_3)e^{m_1x} + \dots + c_ne^{m_nx}$$

Case 3 : Two of the roots of the auxiliary eqn are complex.

Let the two roots of A.E be $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete soln is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$y = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

By Euler's theorem $e^{i\theta} = \cos\theta + i\sin\theta$

$$= e^{\alpha x}(A\cos\beta x + B\sin\beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

which is the required soln.

Case 4 :

Let $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$

The complete soln is

$$y = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + c_5 e^{5x} \dots + c_n e^{m_n x}$$

working Rule to solve the eqn :-

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots K_n y = X$$

of which the eqn is in symbolic form

$$D^n y + K_1 D^{n-1} y + \dots + K_n y = X$$

$$(D^n + K_1 D^{n-1} + \dots + K_n) y = X$$

$$f(D) y = X$$

where $f(D) = (D^n + K_1 D^{n-1} + \dots K_n)$

Step I :

To find the complementary funⁿ

(i) write the A.E $F(D)y = 0$ and solve it. Let it's root be $D = m_1, m_2, \ldots, m_n$.

Write the complete solⁿ

Roots of A.E.

(1) All the roots $m_1, m_2, m_3 \ldots m_n$ are real & different.

(2) All the roots $m_1, m_2, m_3 \ldots m_n$ are real & $m_1 = m_2$.

(3) All the roots $m_1, m_2, m_3 \ldots$ are real & $m_1 = m_2 = m_3$.

(4) Two pairs of the roots are complex i.e $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$ & all other roots are real & different.

Complete Solution :

(i) $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

(ii) $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

(iii) $y = (c_1 x + c_2 x^2 + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$

(iv) $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

(v) $y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}$

Observe the following illustrative table for writing the complementary funⁿ based on the roots of the auxiliary eqⁿ.

Roots of the A.E.	complimentary funⁿ (y_c)
1. 2, 3	$c_1 e^{2x} + c_2 e^{3x}$
2. 1, −1, 2, −2	$c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$
3. 3, 3	$(c_1 + c_2 x) e^{3x}$
4. 0, −2, −2, −2	$c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2) e^{-2x}$
5. $\pm 2i$	$c_1 \cos 2x + c_2 \sin 2x$
6. $3 \pm 2i$	$e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$
7. 1, 1, $1 \pm i$	$(c_1 + c_2 x) e^{x} + e^{x}(c_3 \cos x + c_4 \sin x)$
8. $1 \pm 2i$, $1 \pm 2i$	$e^{x}\{c_1 + c_2 x\} \cos 2x + (c_3 + c_4 x) \sin 2x$

working procedure for problems to solve a homogeneous D.E with constant coefficients.

- The given D.E is put in the form $f(D)\,y = 0$
- form the A.E $F(m) = 0$ and solve the same (Here we need to adopt various techniques to solve the eqn including the synthetic-division method.)
- Based on the nature of the roots of the A.E we write the C.F which itself is the general soln of the D.E taking into account the dependent and the independent variables involved in the D.E.

Roots of the A.E.	complimentary funn (y_c)
1. $2, 3$	$c_1 e^{2x} + c_2 e^{3x}$
2. $1, -1, 2, -2$	$c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$
3. $3, 3$	$(c_1 + c_2 x)\, e^{3x}$
4. $0, -2, -2, -2$	$c_1 e^{0x} + (x_2 + c_3 x + c_4 x^2) e^{-2x}$
5. $\pm 2i$	$c_1 \cos 2x + c_2 \sin 2x$
6. $3 \pm 2i$	$e^{3x} (c_1 \cos 2x + c_2 \sin 2x)$
7. $1, 1, 1 \pm i$	$(c_1 + c_2 x) e^{x} + e^{x} (c_3 \cos x + c_4 \sin x)$
8. $1 \pm 2i, 1 \pm 2i$	$e^{x} \{c_1 + c_2 x\} \cos 2x + (c_3 + c_4 x) \sin 2x$

Solution of Differential Equation $F(D)y = X$ or P.I.

We have already discuss from the previous chapter, that the sol of the equation $f(D)y = X$, consists of two parts, namely complementary function and particular integral. The complementary function for this equation is same as the complete solⁿ of $f(D) = 0$. The methods to find complementary functions have already been discussed. we shall discuss the methods of finding the particular Integrals.

Particular Integral :-

The particular integral of the differential equation $F(D)y = X$ is defind as $y = \frac{1}{F(D)} X$ . . . . . (1)

Where in general 'X' is a function of x, It may of course be constant. The symbol $\frac{1}{F(D)} X$, is defined as a function of x, which when separated upon by $F(D)$, gives X, in - Symbolically $F(D)\left[\frac{1}{F(D)} X\right] = X$.

Thus the function $\frac{1}{F(D)} X$ satisfies the eqⁿ (1) and - called particular integral.

As already pointed, the polynomial $F(D)$, can be subjected to algebraic operations, such as factorisation, resolutions in partial fractions and expansions by binomial theorem etc. The following results are quite useful to find particular integrals.

Results :-

(1) If X is a function of x or a const, then $\frac{1}{D} X = \int X \, dx$

Solⁿ :- Let $y = \frac{1}{D} X$ operating both sides D

$$Dy = D\left(\frac{1}{D} X\right) = X$$

or $\frac{dy}{dx} = X$, Integrating both side w.r.t x $y = \int X dx$, hence the result.

(ii) If X is a function of x or a constant, then $\frac{1}{(D-m)}X = e^{mx}\int X e^{-mx}dx$

Soln: Let $\frac{1}{(D-m)}X = y$

$(D-m)y = (D-m)\left(\frac{1}{(D-m)}X\right) = X$

or $\frac{dy}{dx} - my = X$, which is linear

$I.F = e^{\int -mdx} = e^{-mx}$

Hence soln is $ye^{-mx} = \int X e^{-mx}dx$

$y = e^{mx}\int X e^{-mx}dx$ $\quad \therefore \frac{1}{D-m}X = e^{mx}\int X e^{-mx}dx$

This formula enables us to evalute the particular integral $\frac{1}{F(D)}X$,

where $f(D) = (D-m_1)(D-m_2)\cdots\cdots(D-m_n)$

$\therefore \frac{1}{F(D)}X = \frac{1}{(D-m_1)(D-m_2)\cdots\cdots(D-m_n)}X$

Resolving R.H.S. into partial functions.

$\frac{1}{F(D)}X = \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2}\cdots\cdots + \frac{A_n}{D-m_n}\right)X$

$= A_1 e^{m_1x}\int x e^{-m_1x} + A_2 e^{m_2x}\int X e^{-m_2x}dx + \cdots + A_n e^{m_nx}\int X e^{m_nx}dx$

Particular Integrals in some special cases :

Let us consider the differential eqn.

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots\dots + P_n) y = X$$

where all P_i's are constant.

The above eqn can be written as $F(D) y = X$

particular integral $= \frac{1}{F(D)} X$

Case 1 :

when $X = e^{ax}$

$D(e^{ax}) = a e^{ax}$

$D^2(e^{ax}) = a^2 e^{ax}$

$\dots\dots\dots$

$D^n(e^{ax}) = a^n e^{ax} \quad \dots\dots (i)$

Multiplying in (i) by $P_n, P_{n-1} \dots\dots P_1$ and respectively and adding,

$$(D^n + P_1 D^{n-1} + \dots\dots P_{n-1} D + P_n) e^{ax} = (a^n + P_1 a^{n-1} + \dots\dots P_{n-1} a + P_n) e^{ax}$$

$\Rightarrow F(D) e^{ax} = F(a) e^{ax}$

Operating both sides of eqn (ii) by $\frac{1}{F(D)}$

$$\frac{1}{F(D)} [F(D) e^{ax}] = \frac{1}{F(D)} [F(a) e^{ax}]$$

or $$\frac{e^{ax}}{F(a)} = \frac{1}{F(D)} e^{ax}$$

which gives the particular integral of e^{ax}

In case $X = K$ (a constant) then

$$\frac{1}{F(D)} K = K \frac{1}{F(D)} e^{0x} = \frac{K}{F(D)}, \text{ provided } F(0) \neq 0$$

Case of Failure :- If $F(a)=0$, the above method fails and we proceed as under.

Since $F(a)=0$

$D=a$, is a root of $F(D)=0$ ...... (i)

$(D-a)$ is a factor of $F(D)$, suppose $F(D)=(D-a)F'(D)$.

where $F'(a)\neq 0$.

Then $\frac{1}{F(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{F'(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{F'(a)}e^{ax}$

$= \frac{1}{F'(a)}\cdot\frac{1}{(D-a)}e^{ax} = \frac{1}{F'(a)}e^{ax}\int e^{ax}\cdot e^{-ax}\cdot dx$

$= \frac{1}{F'(a)}e^{ax}\int dx = x\cdot\frac{1}{F'(a)}e^{ax}$

i.e. $\frac{1}{F(D)}e^{ax} = x\frac{1}{F'(a)}e^{ax}$ ...... (2) $(F'(a)\neq 0)$

$$\left[\begin{array}{l} \because F'(D) = (D-a)F''(D)+1.F(D) \\ \because F'(a) = 0\,F''(a)+F(a) \end{array}\right]$$

If $F'(a)=0$, then applying (2) again

we get $\frac{1}{F(D)}e^{ax} = x^2\frac{1}{F''(a)}e^{ax}$ provided $F''(a)\neq 0$ .... (3)

and so on,

In general if $D=a$, is a repeated root of $F(D)=0$, say $(r+1)$ times,

we have $\frac{1}{F(D)}e^{ax} = \frac{x^r\cdot e^{ax}}{F^r(a)}$. $F^r(a)\neq 0$

Rule 1 :-

To find the P.I. $\frac{1}{F(D)} e^{ax}$, replace D by a, provided $F(a) \neq 0$, In case $F(a) = 0$ multiply the numerator by x and differentiate the denominator w.r.t. D and apply the above rule, provided $F'(a) \neq 0$ and so on.

-: Partial Differential Equation :-

* The equation which contain one or more partial derivative is called partial differential equation. They must involve at least two independent variable and one dependent variable whenever we consider the case of two independent variable we shall usually take them to be x and y and z to be the dependent variable.

* The partial differential co-efficents $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ will be denoted by Q and P respectively.

* The second order partial derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ r, s, t respectively.

* The order of a partial derivative or partial differential equation is same as the order of the highest partial - derivative in the equation. And it's degree is the degree of this derivative

for example :-

(i) $x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial x} = z$

here order 1, degree 1

(ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z^2}{\partial x \partial y} + \frac{\partial z^2}{\partial y^2} = 0$

here

* The partial differential eqn the form either by the ellimination of function from a relation involving three of variables.

Consider z to be a function of x and y defined by $F(x, y, z, a, b) = 0$ ——— (1) in which a and b are to constant differenceting equation partially w.r.t x.

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \times \frac{\partial z}{\partial x} = 0$$

$$= \frac{\partial f}{\partial x} + P\frac{\partial F}{\partial z} = 0 \quad \text{——— (2)}$$

differenciating eqⁿ (1) partially w.r.t eqⁿ (2) w.r.t y

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \times \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} = 0 \quad \text{--- (3)}$$

By means of the eqⁿ (1), (2), (3) to constants a and b can be elliminated, this results in a partial differential eqⁿ of order 1 in the form $F(x, y, z, p, q) = 0$

FIND THE SOLUTION OF $F(D)Y = E^{AX}$:-

General solution of $F(D)y = X$ (Non-homogenous linear equation)

Consider the non-homogeneous linear differential eq^n of n^{th} order with co-efficients.

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n)\, y = X$$

where.

$a_0, a_1, \ldots\ldots, a_n$ are rational co-efficients

i.e. $F(D)\, y = X$ ——— ①

where,

$$F(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n.$$

X is a function of x.

→ The complete integral of equation ① consists of two parts namely the complementary function (C.F) & a particular integral (P.I)

The particular integral of eq^n ① is denoted by

$$P.I = \frac{1}{F(D)} \cdot X.$$

Now suppose $F(D) = D - a$

The P.I is given by $y = \dfrac{1}{D-a} X$

$$\Rightarrow (D-a)y = X$$

$$\Rightarrow \frac{dy}{dx} - ay = X$$

which is a linear eq^n in y

Hence the solution is

$$y = \frac{1}{D-a} X = e^{ax} \int e^{-ax} X\, dx.$$

$$\therefore \frac{1}{D-a} X = e^{ax} \int e^{-ax} X \cdot dx \quad \text{———} \quad ②$$

If $F(D) = a_0 D^2 + a_1 D + a_2$ i.e second degree,

then we can express

$$F(D) = (D-\alpha)(D-\beta)$$

So, PI is given by $\frac{x}{(D-\alpha)(D-\beta)}$

Resolve $\frac{1}{(D-\alpha)(D-\beta)}$ into partial fractions and then use the relation eqn (2) to evaluate the P.I.

$\therefore$ General solution is

$$\boxed{y = C.F + P.I}$$

To find P.I. when X is of the form $X = e^{ax}$

The P.I $= \frac{1}{F(D)} X = \frac{1}{F(D)} e^{ax}$

we know that

$$D(e^{ax}) = a e^{ax}$$

$$D^2(e^{ax}) = a^2 e^{ax}$$

$$\vdots$$

$$D^n(e^{ax}) = a^n e^{ax}$$

$$\therefore F(D) e^{ax} = F(a) e^{ax}$$

$$\frac{1}{F(D)}\left[F(D) e^{ax}\right] = \frac{1}{F(D)}\left[F(a) e^{ax}\right] = F(a) \frac{1}{F(D)} e^{ax}$$

$$\frac{1}{F(D)}\left[F(D) e^{ax}\right] = F(a) \frac{1}{F(D)} e^{ax}$$

$$\Rightarrow e^{ax} = F(a) \cdot \frac{e^{ax}}{F(D)}$$

$$\Rightarrow \frac{1}{F(D)} e^{ax} = \frac{e^{ax}}{F(a)}$$ provided $F(a) \neq 0$

Now if

$F(a) = 0$, then $F(D) = (D-a)\, g(D)$ where $g(a) \neq 0$

So,

$$PI = \frac{1}{F(D)} X$$

$$= \frac{1}{F(D)} \cdot e^{ax} = \frac{e^{ax}}{(D-a)\,g(D)} = \frac{1}{g(a)} \frac{e^{ax}}{D-a}$$

$$= \frac{1}{g(a)} \left[e^{ax} \int e^{-ax} \cdot e^{ax}\, dx \right]$$

$$\left[\because \frac{x}{D-a} = e^{ax} \int e^{-ax} X\, dx \right]$$

$$\Rightarrow P.I = \frac{1}{g(a)} \left[e^{ax} \int dx \right] = \frac{x e^{ax}}{g(a)}$$

we can also take
$g(D) = F(D)$ here

Similarly,

$$\frac{e^{ax}}{(D-a)^2} = \frac{x^2 e^{ax}}{2!} \text{ and } \frac{e^{ax}}{(D-a)^3} = \frac{x^3 e^{ax}}{3!}$$

Working rule for evaluating P.I for $F(D)y = e^{ax}$:-

Write the factors of $F(D) = (D-\alpha)(D-\beta)$

Substitute $D = a$ if $F(a) \neq 0$

If $F(a) = 0$, then

$$\frac{1}{F(D)} \cdot e^{ax} = \frac{1}{(D-a)} e^{ax} = \frac{x}{1} e^{ax},$$

$$\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$$

$$\frac{1}{(D-a)^3} e^{ax} = \frac{x^3}{3!} e^{ax}$$

Ex 1 solve $(D^2 - 7D + 6)y = e^{2x}$

Ex 2 solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = e^{2x}$

3. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$

4. Solve $(4D^2 + 4D - 3)y = e^{2x}$

5. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{4x}$

6. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}$

7. Solve $(D^2 + 6D + 5)y = 16e^{3x} + 2e^{-x} + 3$

8. Solve $y'' + 4y' + 5y = -2\cosh x$. Also find y where $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$

Find the solution of $f(D)y = \cos ax$ or $\sin ax$. :-

⇒ Particular Integral of $F(D) = X$ when $X = \cos ax$ or $\sin ax$ where 'a' is any constant.

Let $X = \sin ax$ then

$$P.I = \frac{1}{F(D)}X = \frac{1}{F(D)}\sin ax$$

we have.

$D(\sin ax) = a\cos ax$, $D^2(\sin ax) = -a^2\sin ax$

$D^3(\sin ax) = -a^3\cos ax$, $D^4(\sin ax) = a^4\sin ax$

$D^5(\sin ax) = a^5\cos x$, $D^6(\sin ax) = -a^6\sin ax$ and so on

Hence from the above it is clear that

$D^2(\sin ax) = -a^2\sin ax$

$D^4(\sin ax) = a^4\sin ax = (-a^2)^2\sin ax$

$D^6(\sin ax) = (-a^2)^3\sin ax = -a^6\sin ax$

. . . .

. . . .

$D^{2n}(\sin ax) = (D^2)^n\sin ax = (-a^2)^n\sin ax$

Since $F(D) = D^n + a_1D^{n-1} - a_2D^{n-2} + \dots + a_n$

$\therefore F(D^2)\sin ax = \left[(D^2)^n + a_1(D^2)^{n-1} + a_2(D^2)^{n-2} + \ldots + a_n\right]\sin ax$

$= \left[(-a^2)^n \sin ax + a_1(-a^2)^{n-1}\sin ax + a_2(-a^2)^{n-2}\sin ax + \ldots \right.$

$\left. \ldots + a_n \sin ax \right.$

$= \left[(-a^2)^n + a_1(-a^2)^{n-1} + a_2(-a^2)^{n-2} + \ldots + a_n\right]\sin ax$

$\Rightarrow F(D^2)\sin ax = F(-a^2)\sin ax$

Operating on both sides by $\frac{1}{F(D^2)}$, we get

$$\frac{F(D^2)\sin ax}{F(D^2)} = \frac{F(-a^2)\sin ax}{F(D^2)} \quad \because F(-a^2) \text{ is a const.}$$

$$\Rightarrow \sin ax = F(-a^2)\cdot\frac{1}{F(D^2)}\sin ax$$

$$\frac{F(D^2)\sin ax}{F(D^2)} = \frac{F(-a^2)\sin ax}{F(D^2)} \quad \because F(-a^2) \text{ is a const.}$$

$$\Rightarrow \sin ax = F(-a^2)\frac{1}{F(D^2)}\sin ax$$

$$\Rightarrow \frac{1}{F(D^2)}\sin(ax) \frac{1}{F(-a^2)}\sin ax, \text{ if } F(-a^2) \neq 0$$

Hence $P.I = \frac{1}{F(D^2)}\sin ax$, if $F(-a^2) \neq 0$

$= \frac{1}{F(-a^2)}\sin$