GOVERNMENT POLYTECHNIC DHENKANAL

LECTURE

NOTES ON

ENGINEERING MATHEMATICS - III

3rd SEMESTR

PREPARED BY
SATABDIKA NAYAK

MATRICES

After studying this chapter you will acquire the skills in

- knowledge on matrices
- Knowledge on matrix operations.
- Matrix as a tool of solving linear equations with two or three unknowns.

List of References:

- Frank Ayres, JR, Theory and Problems of Matrices Sohaum's Outline Series
- Datta KB , Matrix and Linear Algebra
- Vatssa BS, Theory of Matrices, second Revise Edition
- Cooray TMJA, Advance Mathematics for Engineers, Chapter 1- 4

Chapter I: Introduction of Matrices

1.1 Definition 1:

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A,B, C* etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

A is a matrix of order $m \times n$. i^{th} row j^{th} column element of the matrix denoted by a_{ij}

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

1.2 Special Types of Matrices:

1. **Square matrix**:

A matrix in which numbers of rows are equal to number of columns is called a square matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}$$

2. **Diagonal matrix**:

A square matrix $A = (a_{ij})_{n \times n}$ is called a diagonal matrix if each of its non-diagonal element is zero.

That is $a_{ij} = 0 \;\; if \; i \neq j$ and at least one element $a_{ii} \neq 0$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

3. **Identity Matrix**

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n.

That is $a_{ij} = \begin{cases} 0 & if \; i \neq j \\ 1 & if \; i = j \end{cases}$

Example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. **Upper Triangular matrix**:

A square matrix said to be a Upper triangular matrix if $a_{ij} = 0 \;\; if \; i > j$.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 8 \\ 0 & -2 & 5 \\ 0 & 0 & 7 \end{pmatrix}$$

5. **Lower Triangular Matrix**:

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0 \;\; if \; i < j$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 0 & 0 \\ 9 & 6 & 2 \end{pmatrix}$$

6. **Symmetric Matrix**:

A square matrix $A = (a_{ij})_{n \times n}$ said to be a symmetric if $a_{ij} = a_{ji}$ for all *i* and *j*.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ -2 & -9 & 3 \\ 7 & 3 & 5 \end{pmatrix}$$

7. Skew- Symmetric Matrix:

A square matrix $A = (a_{ij})_{n\times n}$ said to be a skew-symmetric if $a_{ij} = -a_{ji}$ for all *i* and *j.*

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ 2 & -9 & 3 \\ -7 & -3 & 5 \end{pmatrix}$$

8. Zero Matrix:

A matrix whose all elements are zero is called as Zero Matrix and order $n \times m$ Zero matrix denoted by $0_{n\times m}$.

Example:

$$0_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. Row Vector

A matrix consists a single row is called as a row vector or row matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \qquad B = \begin{pmatrix} 7 & 4 & -3 \end{pmatrix}$$

10. Column Vector

A matrix consists a single column is called a column vector or column matrix.

Example:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad B = \begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix}$$

Chapter 2: Matrix Algebra

2.1. Equality of two matrices:

Two matrices A and B are said to be equal if

(i) They are of same order.
(ii) Their corresponding elements are equal.

That is if $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ then $a_{ij} = b_{ij}$ for all *i* and *j.*

2.2. Scalar multiple of a matrix

Let k be a scalar then scalar product of matrix $A = (a_{ij})_{m \times n}$ given denoted by *kA* and given by $kA = (ka_{ij})_{m \times n}$ or

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ \vdots & & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

2.3. Addition of two matrices:

Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ are two matrices with same order then sum of the two matrices are given by

$$A + B = (a_{ij})_{m \times n} + (b_{ij})_{m \times n} = (a_{ij} + b_{ij})_{m \times n}$$

Example 2.1: let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 8 \end{pmatrix}.$$

Find (i) 5B (ii) A + B (iii) 4A – 2B (iv) 0 A

2.4. Multiplication of two matrices:

Two matrices *A* and *B* are said to be confirmable for product *AB* if number of columns in *A* equals to the number of rows in matrix *B*. Let $A = (a_{ij})_{m \times n}$ $and\ B = (b_{ij})_{n \times r}$ be two matrices the product matrix *C*= *AB*, is matrix of order $m \times r$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots\dots + a_{in}b_{nj}$$

Example 2.2: Let $A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{pmatrix}$

Calculate (i) AB (ii) BA

(iii) is AB = BA ?

2.5. Integral power of Matrices:

Let A be a square matrix of order n, and m be positive integer then we define

$$A^m = A \times A \times A \dots\dots \times A \quad \text{(m times multiplication)}$$

2.6. Properties of the Matrices

Let A, B and C are three matrices and λ and μ are scalars then

(i) $A + (B + C) = (A + B) + C$ Associative Law

(ii) $\lambda(A+B) = \lambda A + \lambda B$ Distributive law

(iii) $\lambda(\mu A) = (\lambda\mu)A$ Associative Law

(iv) $(\lambda A)B = \lambda(AB)$ Associative Law

(v) $A(BC) = (AB)C$ Associative Law

(vi) $A(B+C) = AB + AC$ Distributive law

2.7. Transpose:

The transpose of matrix $A = (a_{ij})_{m\times n}$, written A^t (A' or A^T) is the matrix obtained by writing the rows of A in order as columns.

That is $A^t = (a_{ji})_{n\times m}$.

Properties of Transpose:

(i) $(A+B)^t = (A^t + B^t)$
(ii) $(A^t)^t = A$
(iii) $(kA)^t = kA^t$ for scalar k.
(iv) $(AB)^t = B^t A^t$

Example 2.3: Using the following matrices A and B, Verify the transpose properties

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

Proof: (i) Let a_{ij} and b_{ij} are the $(i,j)^{th}$element of the matrix A and B respectively. Then $a_{ij} + b_{ij}$ is the $(i,j)^{th}$ element of matrix $A+B$ and it is $(j,i)^{th}$ element of the matrix $(A+B)^t$

Also a_{ij} and b_{ij} are the $(j,i)^{th}$element of the matrix A^t and B^trespectively. Therefore $a_{ij} + b_{ij}$ is the $(j,i)^{th}$ element of the matrix $A^t + B^t$

(ii) Let $(i,j)^{th}$ element of the matrix A is a_{ij}, it is $(j,i)^{th}$ element of the A^t then it is $(i,j)^{th}$ element of the matrix $(A^t)^t$

(iii) try

(iv) $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots + a_{in}b_{nj}$ is the $(i,k)^{th}$ element of the AB It is result of the multiplication of the i^{th} row and k^{th} column and it is $(k,i)^{th}$ element of the matrix $(AB)^t$.

$B^t A^t$, $(k,i)^{th}$ element is the multiplication of k^{th} row of $B^t$with i^{th} column of A^t, That is k^{th} column of B with i^{th} row of A.

2.8 A square matrix A is said to be symmetric if $A = A^t$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$, A is symmetric by the definition of symmetric matrix.

Then

$$A^t = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$

That is $A = A^t$

2.9 A square matrix A is said to be skew- symmetric if $A = -A^t$

Example:

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -5 & 8 \\ 1 & 8 & 9 \end{pmatrix}$$

(i) AA^t and A^tA are both symmetric.

(ii) $A + A^t$ is a symmetric matrix.

(iii) $A - A^t$ is a skew-symmetric matrix.

(iv) If A is a symmetric matrix and m is any positive integer then A^m is also symmetric.

(v) If A is skew symmetric matrix then odd integral powers of A is skew symmetric, while positive even integral powers of A is symmetric.

If *A* and *B* are symmetric matrices then

(vi) $(AB + BA)$ is symmetric.

(vii) $(AB - BA)$ is skew-symmetric.

Exercise 2.1: Verify the (i) , (ii) and (iii) using the following matrix A.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ -3 & -5 & 10 \\ 1 & 8 & 9 \end{pmatrix}$$

Chapter 3: Determinant, Minor and Adjoint Matrices

Definition 3.1:

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n , then the number $|A|$ called determinant of the matrix A.

(i) Determinant of 2×2 matrix

Let A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|A|$ = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}a_{22} - a_{12}a_{21}$

(ii) Determinant of 3 × 3 matrix

Let B = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then $|B|$ =

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Exercise 3.1: Calculate the determinants of the following matrices

(i) $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

3.1 Properties of the Determinant:

a. The determinant of a matrix A and its transpose A^t are equal.

$$|A| = |A^t|$$

b. Let A be a square matrix

(i) If A has a row (column) of zeros then $|A| = 0$.

(ii) If A has two identical rows (or columns) then $|A| = 0$.

c. If A is triangular matrix then $|A|$ is product of the diagonal elements.

d. If A is a square matrix of order n and k is a scalar then $|kA| = k^n|A|$

3.2 Singular Matrix

If *A* is square matrix of order n, the *A* is called singular matrix when $|A| = 0$ and non-singular otherwise.

3.3. Minor and Cofactors:

Let A = $(a_{ij})_{n \times n}$is a square matrix. Then M_{ij} denote a sub matrix of A with order (n-1) × (n-1) obtained by deleting its i^{th}row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} denoted by A_{ij} and is equal to$(-1)^{i+j}|M_{ij}|$.

Exercise 3.2: Let $A = \begin{pmatrix} 5 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & -2 & -1 \end{pmatrix}$

(i) Compute determinant of A.

(ii) Find the cofactor matrix.

3.4. Adjoin Matrix:

The transpose of the matrix of cofactors of the element a_{ij} of A denoted by $adj\ A$ is called adjoin of matrix A.

Example 3.3: Find the adjoin matrix of the above example.

Theorem 3.1:

For any square matrix A,

$$A\ (adj\ A\ ) = (adj\ A\ )A = \ |A|\ I$$ where I is the identity matrix of same order.

Proof: Let A = $(a_{ij})_{n\times n}$

Since A is a square matrix of order n, then $adj\ A$ also in same order.

Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$ then

$$adj\ A = \begin{pmatrix} A_{11} & A_{12} & \dots\dots & A_{1n} \\ A_{21} & A_{22} & \dots\dots & A_{2n} \\ \dots & \dots & \dots\dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

Now consider the product $A\ (adj\ A\ )$

$$A\ (adj\ A\ ) = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \dots\dots & A_{1n} \\ A_{21} & A_{22} & \dots\dots & A_{2n} \\ \dots & \dots & \dots\dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}\, A_{1j} & \sum_{j=1}^{n} a_{1j}\, A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{1j}\, A_{nj} \\ \sum_{j=1}^{n} a_{2j}\, A_{1j} & \sum_{j=1}^{n} a_{2j}\, A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{2j}\, A_{nj} \\ \dots\dots & \dots\dots & \dots\dots & \dots \\ \sum_{j=1}^{n} a_{nj}\, A_{1j} & \sum_{j=1}^{n} a_{nj}\, A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{nj}\, A_{nj} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots\dots & |A| \end{pmatrix}$$

(as we know that $\sum_{j=1}^{n} a_{ij}\, A_{ij} \ = \ |A|$ and $\sum_{j=1}^{n} a_{ij}\, A_{kj} = 0$ when $i \neq k$)

$$= |A| \begin{pmatrix} 1 & 0 & \dots.. & 0 \\ 0 & 1 & \dots.. & 0 \\ 0 & 0 & \dots.. & 0 \\ 0 & 0 & \dots\dots & 1 \end{pmatrix}$$

$= |A| \, I_n$ Where I_n is unit matrix of order n.

Theorem 3.2: If A is a non-singular matrix of order n, then $|adj\ A| = |A|^{n-1}$.

Proof: By the theorem 1

$$A\,(adj\ A) = |A|\ I$$

$$|A\,(adj\ A)| = ||A|\ I|$$

$$|A||\ adj\ A\ | = |A|^n$$

$$|\ adj\ A\ | = |A|^{n-1}$$

Theorem 3.3: If A and B are two square matrices of order n then

$$adj(AB) = (adj\ B)(adj\ A)$$

Proof: By the theorem 1 $A\,(adj\ A) = |A|\ I$

Therefore $(AB)\ adj\ (AB) = adj\ (AB)AB = |AB|I$

Consider $(AB)(adj\ B\ adj\ A)$,

$$(AB)(adj\ B\ adj\ A) = A\,(B\ adj\ B)\ adj\ A$$

$$= A(|B|\ I)\ adj\ A$$

$$= |B|\ (A\ adj\ A)$$

$$= |B|\ |A|\ I$$

$$= |A||B|\ I$$

$= |AB|I$ ……................... (i)

Also consider $(adj\ B\ .\ adj\ A)A\,B$

$$(adj\ B\ .\ adj\ A)A\,B = adj\ B\ (adj\ A\ A)B$$

$$= adj\ B\ |A|\ I\ B$$

$$= |A| adj\ B\ B$$

$$= |A||B|\ I$$

$= |AB|I$ …………….. (ii)

Therefore from (i) and (ii) we conclude that

$$adj(AB) = (adj\ A)(adj\ B)$$

Some results of adjoint

(i) For any square matrix A $(adj\ A)^t = adj\ A^t$

(ii) The adjoint of an identity matrix is the identity matrix.

(iii) The adjoint of a symmetric matrix is a symmetric matrix.

.Chapter 4: Inverse of a Matrix and Elementary Row Operations

4.1 Inverse of a Matrix

Definition 4.1:

If A and B are two matrices such that $AB = BA = I$, then each is said to be inverse of the other. The inverse of A is denoted by A^{-1}.

Theorem 4.1: (Existence of the Inverse)

The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$ (That is A is non singular).

Proof: (i) The necessary condition

Let A be a square matrix of order n and B is inverse of it, then

$$AB = I$$

$$|AB| = |A||B| = I$$

Therefore $|A| \neq 0$.

(ii) The sufficient condition:

If $|A| \neq 0$, the we define the matrix B such that

$$B = \frac{1}{|A|}\ (adj\ A)$$

Then $AB = A\,\frac{1}{|A|}\ (adj\ A) = \frac{1}{|A|}\ A(adj\ A)$

$$= \frac{1}{|A|}\ |A|I = I$$

Similarly $BA = \frac{1}{|A|}\ (adj\ A)A = \frac{1}{|A|}\ A(adj\ A) = \frac{1}{|A|}\ |A|I = I$

Thus $AB = BA = I$ hence B is inverse of A and is given by $A^{-1} = \frac{1}{|A|}\ (adj\ A)$

Theorem 4.2: (Uniqueness of the Inverse)

Inverse of a matrix if it exists is unique.

Proof: Let B and C are inverse s of the matrix A then

$$AB = BA = I \text{ and } AC = CA = I$$

$$B(AC) = BI$$

$$(BA)C = B$$

$$C = B$$

Example 6: Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ find A^{-1}

Theorem 4.3: (Reversal law of the inverse of product)

If A and B are two non-singular matrices of order n, then (AB) is also non singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

Since A and B are non-singular $|A| \neq 0$ and $|B| \neq 0$, therefore $|A||B| \neq 0$, then $|AB| \neq 0$.

Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AIA^{-1} = AA^{-1} = I$ ....(1)

Similarly $(B^{-1}A^{-1})\ (AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}IB = B^{-1}B = I$ ....(2)

From (1) and (2)

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})\ (AB) = I$$

Therefore by the definition and uniqueness of the inverse $(AB)^{-1} = B^{-1}A^{-1}$

Corollary4.1: If $A_1A_2 \dots\dots\dots A_m$ are non singular matrices of order n, then $(A_1A_2 \dots\dots\dots A_m)^{-1} = A_1^{-1}A_2^{-2} \dots\dots\dots A_m^{-1}$.

Theorem 4.4: If A is a non-singular matrix of order n then $(A^t)^{-1} = (A^{-1})^t$.

Proof: Since $|A^t| = |A| \neq 0$therefore the matrix A^t is non-singular and $(A^t)^{-1}$ exists.

Let $AA^{-1} = A^{-1}A = I$

Taking transpose on both sides we get

$$(AA^{-1})^t = (A^{-1})^tA^t = I^t{}_n = I_n$$

$$(A^{-1}A)^t = A^t(A^{-1})^t = I^t = I_n$$

Therefore $A^t(A^{-1})^t = (A^{-1})^tA^t = I_n$

That is $(A^{-1})^t = = (A^t)^{-1}$.

Theorem 4.5: If A is a non-singular matrix , k is non zero scalar, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Proof: Since A is non-singular matrix A^{-1} exits.

Let consider $(kA) \left(\frac{1}{k}A^{-1}\right) = \left(k \times \frac{1}{k}\right) (A\, A^{-1}) = I$

Therefore $\left(\frac{1}{k}A^{-1}\right)$ is inverse of kA

By uniqueness if inverse $(kA)^{-1} = \frac{1}{k}A^{-1}$

Theorem 4.6: If A is a non-singular matrix then

$$|A^{-1}| = \frac{1}{|A|}.$$

Proof: Since A is non-singular matrix, A^{-1} exits and we have

$$AA^{-1} = I$$

Therefore $|AA^{-1}| = |A||A^{-1}| = |I| = 1$

Then $|A^{-1}| = \frac{1}{|A|}$

4.2 Elementary Transformations:

Some operations on matrices called as elementary transformations. There are six types of elementary transformations, three of then are row transformations and other three of them are column transformations. There are as follows

(i) Interchange of any two rows or columns.
(ii) Multiplication of the elements of any row (or column) by a non zero number k.
(iii) Multiplication to elements of any row or column by a scalar k and addition of it to the corresponding elements of any other row or column.

We adopt the following notations for above transformations

(i) Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$.
(ii) Multiplication by k to all elements in the i^{th} row $R_i \rightarrow kR_i$.
(iii) Multiplication to elements of j^{th} row by k and adding them to the corresponding elements of i^{th} row is denoted by $R_i \rightarrow R_i + kR_j$.

4.2.1 Equivalent Matrix:

A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by for forming finitely many successive elementary transformations on a matrix A.

Denoted by A~ B.

4.3 Rank of a Matrix:

Definition:

A positive integer 'r' is said to be the rank of a non- zero matrix A if

(i) There exists at least one non-zero minor of order r of A and
(ii) Every minor of order greater than r of A is zero.

The rank of a matrix A is denoted by $\rho(A)$.

4.4 Echelon Matrices:

Definition 4.3:

A matrix $A = (a_{ij})$ is said to be echelon form (echelon matrix) if the number of zeros preceding the first non zero entry of a row increasing by row until zero rows remain.

In particular, an echelon matrix is called a row reduced echelon matrix if the distinguished elements are

(i) The only non- zero elements in their respective columns.
(ii) Each equal to 1.

Remark: The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Example 4.1:

Reduce following matrices to row reduce echelon form

(i) $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix}$

Chapter 5: Solution of System of Linear Equation by Matrix Method

5.1 Solution of the linear system *AX= B*

We now study how to find the solution of system of m linear equations in n unknowns.

Consider the system of equations in unknowns $x_1, x_2, \dots\dots\dots\dots\dots\dots, x_n$ as

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots\dots\dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots\dots\dots a_{2n}x_n = b_2$$

…………………………………………………….

$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\dots\dots\dots a_{nn}x_n = b_n$$

is called system of linear equations with n unknowns $x_1, x_2, \dots\dots\dots\dots\dots, x_n$. If the constants $b_1, b_2 \dots\dots\dots\dots\dots b_n$ are all zero then the system is said to be homogeneous type.

The above system can be put in the matrix form as

$$AX = B$$

Where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$

The matrix $A = (a_{ij})_{n\times n}$ is called coefficient matrix, the matrix X is called matrix of unknowns and B is called as matrix of constants, matrices X and B are of order $n \times 1$.

Definition 5.1: (**consistent**)

A set of values of $x_1, x_2, \dots\dots\dots\dots\dots, x_n$ which satisfy all these equations simultaneously is called the solution of the system. If the system has at least one solution then the equations are said to be consistent otherwise they are said to be inconsistent.

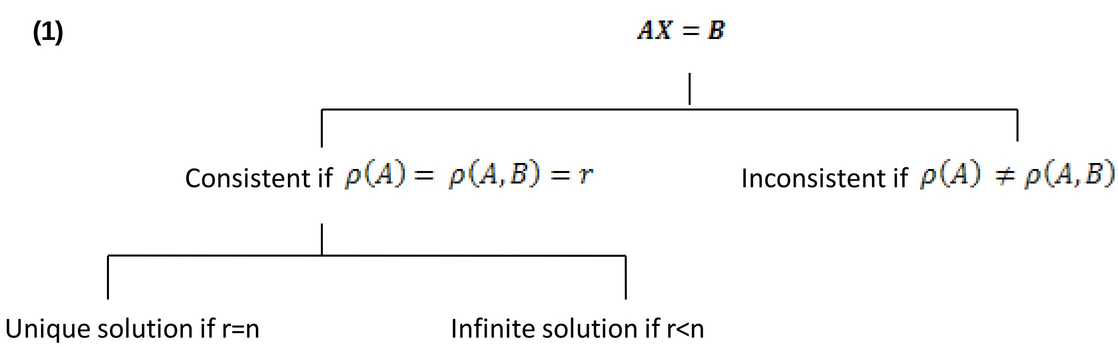
Theorem 5.2:

A system of m equations in n unknowns represented by the matrix equation AX= *B* is consistent if and only if $\rho(A) = \rho(A, B)$. That is the rank of matrix A is equal to rank of augment matrix (A, B)

Theorem 5.2:

If A be an non-singular matrix, X be an $n \times 1$ matrix and B be an $n \times 1$ *m*atrix then the system of equations *AX*= *B* has a unique solution.

(1)



(2)

$\boldsymbol{AX = 0}$ (All wayes consistant)

If $\rho(A) = r = n$ Trivial solution

If $\rho(A) = r < n$ Infinite solutions

Therefore every system of linear equations solutions under one of the following:

(i) There is no solution
(ii) There is a unique solution
(iii) There are more than one solution

Methods of solving system of linear Equations:

5.1 Method of inversesion:

Consider the matrix equation

Consider the matrix equation

$AX = B$ Where $|A| \neq 0$

Pre multiplying by A^{-1}, we have

$$A^{-1}(AX) = A^{-1}B$$

$$X = A^{-1}B$$

Thus $AX = B$, has only one solution if $|A| \neq 0$ and is given by $X = A^{-1}B$.

5.2 Using Elementary row operations: (Gaussian Elimination)

Suppose the coefficient matrix is of the type $m \times n$. That is we have m equations in n unknowns Write matrix $[A, B]$ and reduce it to Echelon augmented form by applying elementary row transformations only.

Example 5.1: Solve the following system of linear equations using matrix method

(i)

$$2x + y + -2z = 10$$
$$y + 10z = -28$$
$$3y + 16z = -42$$

(ii)

$$x + 2y - 3z = -1$$
$$3x - y + 2z = 7$$
$$5x + 3y - 4z = 2$$

Example 5.2: Determine the values of a so that the following system in unknowns x, y and z has

(i) No solutions
(ii) More than one solutions
(iii) A unique solution

$$x + y + z = 0$$

$$2x + 3y + az = 0$$

$$x + ay + 3z = 0$$

Chapter 6: Eigen values and Eigenvectors:

If A is a square matrix of order n and X is a vector in R^n, (X considered as $n \times 1$ column matrix), we are going to study the properties of non-zero X, where AX are scalar multiples of one another. Such vectors arise naturally in the study of vibrations, electrical systems, genetics, chemical reactions, quantum mechanics, economics and geometry.

Definition 6.1:

If A is a square matrix of order n , then a non-zero vector X in $R^n$is called eigenvector of A if $AX = \lambda X$ for some scalar λ. The scalar λ is called an eigenvalue of A, and X is said to be an eigenvector of A corresponding to λ.

Remark: Eigen values are also called proper values or characteristic values.

Example 6.1: The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A= $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Theorem 6.1: If A is a square matrix of order n and λ is a real number, then λ is an eigenvalue of A if and only if $| \lambda I - A | = 0$.

Proof: If λ is an eigenvalue of A, the there exist a non-zero X a vector in R^n such that $\boldsymbol{AX} = \lambda X$.

$$\boldsymbol{AX} = \lambda X$$

$\boldsymbol{AX} = \lambda IX$ Where **I** is a identity matrix of order n.

$$(\lambda I - A)X = 0$$

The equation has trivial solution when if and only if $|A| = 0$. The equation has non-zero solution if and only if $|(A - \lambda I)|$ = 0.

Conversely , if $|(A - \lambda I)|$ = 0 then by the result there will be a non-zero solution for the equation,

$$(A - \lambda I)X = 0$$

That is, there will a non-zero X in R^n such that $AX = \lambda X$, which shows that λ is an eigenvalue of A.

Example 6.2: Find the eigen values of the matrixes

(i) A = $\begin{pmatrix} 2 & 7 \\ 1 & -2 \end{pmatrix}$ (ii) B = $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$

Theorem 6.2:

If A is an $n \times n$ matrix and λ is a real number, then the following are equivalent:

(i) λ is an eigenvalue of A.
(ii) The system of equations $(\lambda I - A)X = 0$ has non-trivial solutions.
(iii) There is a non-zero vector X in R^n such that $AX = \lambda I$.
(iv) Is a solution of the characteristic equation $|(A - \lambda I)|$ = 0.

Definition 6.2:

Let A be an $n \times n$ matrix and λ be the eigen value of A. The set of all vectors X in R^n which satisfy the identity $\boldsymbol{AX} = \lambda X$ is called the eigen space of a corresponding to λ. This is denoted by $E(\lambda)$.

Remark:

The eigenvectors of A corresponding to an eigen value λ are the non-zero vectors of X that satisfy $AX = \lambda I$. Equivalently the eigen vectors corresponding to λ are the non zero in the solution space of $(\lambda I - A)X = 0$. Therefore, the eigen space is the set of all non-zero X that satisfy $(A - \lambda I)X = 0$ with trivial solution in addition.

Steps to obtain eigen values and eigen vectors

Step I : For all real numbers λ form the matrix $\lambda I - A$

Step II: Evaluate $|(A - \lambda I)|$ That is characteristic polynomial of *A*.

Step III: Consider the equation $|(A - \lambda I)| = 0$ (The characteristic equation of *A*) Solve the equation for Let $\lambda_1, \lambda_2, \lambda_3 \ldots \ldots \ldots \ldots \lambda_n$ be eigen values of A thus calculated.

Step IV: For each λ_i consider the equation $(\lambda_i I - A)X = 0$

Find the solution space of this system which an eigen space $E(\lambda_i)$ of A, corresponding to the eigen value λ_i of A . Repeat this for each λ_i $i = 1, 2, \ldots \ldots . n$

Step V: From step IV , we can find basis and dimension for each eigen space $E(\lambda_i)$ for $i = 1, 2, \ldots \ldots . n$

Example 6.3:

Find (i) Characteristic polynomial

(ii) Eigen values
(iii) Basis for the eigen space of a matrix

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

Example 6.4:

Find eigen values of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

Also eigen space corresponding to each value of A. Further find basis and dimension for the same.

6.2 Diagonalization:

Definition 6.2.1: A square matrix *A* is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, the matrix P is said to diagonalizable A.

Theorem 6.2.1: If A is a square matrix of order n, then the following are equivalent.

(i) A is diagonizible.
(ii) A has n linearly independent eigenvectors.

Procedure for diagonalizing a matrix

Step I: Find n linearly independent eigenvectors of A, say $P_1, P_2, \dots \dots \dots .. P_n$

Step II: From the matrix P having $P_1, P_2, \dots \dots \dots .. P_n$ as its column vectors.

Step III: The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2 \dots \dots .. \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to P_i, $i = 1, 2, \dots \dots \dots .. n$.

Example 6.3: Find a matrix P that diagonalizes

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

Tutorial (Matrices)

Q1. Show that the square matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{pmatrix}$ is a singular matrix.

Q2. If $A = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 2 & 5 \\ 1 & 7 & 0 \end{pmatrix}$ determine (i) $|A|$ (ii) *Adj A*

Q3. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{pmatrix}$

Q4. If $A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ determine

(i) B^{-1} (ii) AB (iii) $B^{-1}A$

Q5. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$

(i) Compute $|A|$ (ii) find adj A

(iii) Verify $A(adj\ A) = |A|$ I (iv) Find A^{-1}

Q6. Find the possible value of x can take, given that

$A = \begin{pmatrix} x^2 & 3 \\ 1 & 3x \end{pmatrix}$ $B = \begin{pmatrix} 3 & 6 \\ 2 & x \end{pmatrix}$ such that $AB = BA$.

Q7. If $A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$ find the values of m and n given that $A^2 = mA + nA$

Q8. Find the echelon form of matrix:

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \end{pmatrix}$ Hence discuss (1) unique solution (ii) many solutions and (iii) No solutions of the following system and solve completely.

$x + y + z = 1$

$2x + 3y + 4z = 5$

$4x + 9y + 16z = 25$

Q9. If matrix A is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{pmatrix}$ and I, the unit matrix of order 3, show that $A^3 = pI + qA + rA^2$.

Q10. Let A be a square matrix

a. Show that
$$(I - A)^{-1} = 1 + A + A^2 + A^3 \quad if \quad A^4 = 0$$

b. Show that
$$(I - A)^{-1} = 1 + A + A^2 + A^3 + \cdots \ldots . A^n \quad if \quad A^{n+1} = 0$$

Q11. Find values of a,b and c so that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points (1,2), (-1,6) and (2,3).

Q12. Find values of a,b and c so that the graph of the polynomial $p(x) = ax^2 + bx + c$passes through the points (-1,0) and has a horizontal tangent at (2,-9).

Q13. Let $\begin{pmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{pmatrix}$ be the augmented matrix for a linear- system. For what value of a and b does the system have

a. a unique solution
b. a one- parameter solution
c. a two parameter- solution
d. no solution

Q14. Find a matrix K such that $AKB = C$ given that

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{pmatrix}$$

a. For the triangle below, use trigonometry to show

$$b \cos\gamma \ + \ c \cos\beta \ = a$$

$$c \cos\alpha \ + \ a \cos\gamma \ = b$$

$$a \cos\beta \ + \ b \cos\alpha \ = c$$

And then apply Crame's Rule to show

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Use the Cramer's rule to obtain similar formulas for $\cos\beta$ and $\cos\gamma$.

DIFFERENTIAL EQUATIONS

1. INTRODUCTION

An equation containing an independent variable, dependent variable and differential coefficients is called a differential equation.

(i) $\frac{dy}{dx} = \sin x$ (ii) $\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^3 = 0$ (iii) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} = 5\cos 3x$

2. ORDER OF DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of the above mentioned differential equations are 1, 2, and 4 respectively.

3. DEGREE OF DIFFERENTIAL EQUATION

The degree of a differential equation is the degree of the highest order derivative when differential coefficients are free from radicals and fractions. For example the degrees of above differential equations are 1, 2, and 3 respectively.

Table 24.1: Degree of differential equation

Differential Equation	Order of D.E.	Degree of D.E
$\frac{dy}{dx} + 4y = \sin x$	1	1
$\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 - y = e^x$	2	4
$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = \cos x$	2	1
$\frac{dy}{dx} = \frac{x^4 - y^4}{xy\left(x^2 + y^2\right)}$	1	1

Differential Equation	Order of D.E.	Degree of D.E
$y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$ $\Rightarrow (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - b^2) = 0$	1	2
$\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$	2	2

4. CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Differential equations are first classified according to their order. First-order differential equations are those in which only the first order derivative, and no higher order derivatives appear. Differential equations of order two or more are referred to as higher order differential equations.

A differential equation is said to be linear if the unknown function, together with all of its derivatives, appears in the differential equations with a power not greater than one and not as products either. A nonlinear differential equation is a differential equation which is not linear.

e.g. $y' + y = 0$ is a linear differential equation,

$y'' + yy' + y^2 = 0$ is a non linear differential equation,

Procedure to form a differential equation that represents a given family of curves

Case I:

If the given family F1 of curves depends on only one parameter then it is represented by an equation of the form $F1(x, y, a) = 0$ ... (i)

For example, the family of parabolas $y^2 = ax$ can be represented by an equation of the form

$f(x, y, a): y^2 = ax$

Differentiating equation (i) with respect to x, we get an equation involving y′, y, x and a.

$g(x, y, y', a) = 0$ ... (ii)

The required differential equation is then obtained by eliminating a from equation (i) and (ii) as

$F(x, y, y') = 0$ ... (iii)

Case II:

If the given family F2 of curves depends on the parameters a, b (say) then it is represented by an equation of the form $F2(x, y, a, b) = 0$ ... (iv)

Differentiating equation (iv) with respect to x, we get an equation involving y′, x, y, a, b.

$g(x, y, y', a, b) = 0$ ... (v)

Now we need another equation to eliminate both a and b. This equation is obtained by differentiating equation (v), wrt x, to obtain a relation of the form $h(x, y, y', y'', a, b) = 0$ ... (vi)

The required differential equation is then obtained by elimination a and b from equations (iv), (v) and (vi) as $F(x, y, y', y'') = 0$ ... (vii)

Note: The order of a differential equation representing a family of curves is the same as the number of arbitrary constants present in the equation corresponding to the family of curves.

5. FORMATION OF DIFFERENTIAL EQUATIONS

If an equation is dependent and dependent variables having some arbitrary constant are given, then the differential equation is obtained as follows:

(a) Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) Hence on eliminating arbitrary constants results a differential equation which involves x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ .... $\frac{d^my}{d_xm}$ (where m=number of arbitrary constants).

Illustration 1: Form the differential equation corresponding to $y^2 = m(a^2 - x^2)$, where m and a are arbitrary constants. **(JEE MAIN)**

Sol: Since the given equation contains two arbitrary constant, we shall differentiate it two times with respect to x and we get a differential equation of second order.

We are given that $y^2 = m(a^2 - x^2)$ ... (i)

Differentiating both sides of (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = m(-2x) \Rightarrow y\frac{dy}{dx} = -mx \qquad \text{... (ii)}$$

Differentiating both sides of (ii) w.r.t. to x, we get $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m$ ... (iii)

From (ii) and (iii), we get, $x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$

This is the required differential equation.

Illustration 2: Form diff. equation of $ax^2 + by^2 = 1$ **(JEE MAIN)**

Sol: Similar to the above problem the given equation contains two arbitrary constants, so we shall differentiate it two times with respect to x and then by eliminating a and b we get the differential equation of second order.

$$ax^2 + by^2 = 1 \Rightarrow \quad 2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \quad a + b\,(yy'' + (y')^2) = 0$$

Eliminating a and b we get $\frac{y}{x}y' = yy'' + (y')^2 \quad \Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx} = 0$

Illustration 3: Form the differential equation corresponding to $y^2 = a(b^2 - x^2)$, where a and b are arbitrary constants. **(JEE MAIN)**

Sol: Similar to illustration 1.

We have, $y^2 = a(b^2 - x^2)$ ... (i)

In this equation, there are two arbitrary constants a, b, so we have to differentiate twice, Differentiating the given equation (i) w.r.t. 'x'. We get $2y\frac{dy}{dx} = -2x.a \Rightarrow y\frac{dy}{dx} = -ax$ ... (ii)

Differentiating (ii) with respect to x, we get $y\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx} = -a \Rightarrow y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = -a$ ... (iii)

Substituting the value of a in (ii), we get

$$y\frac{dy}{dx} = \left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}x \Rightarrow y\frac{dy}{dx} = xy\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 \Rightarrow xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

Illustration 4: Find the differential equation of the following family of curves: $xy = Ae^x + Be^{-x} + x^2$ **(JEE MAIN)**

Sol: Here in this problem A and B are the two arbitrary constants, hence we shall differentiate it two times with respect to x and then by eliminating constant terms we will get the required differential equation.

Given: $xy = Ae^x + Be^{-x} + x^2$ ... (i)

Differentiating (i) with respect to 'x', we get $x\frac{dy}{dx} + y = Ae^x - Be^{-x} + 2x$

Again differentiating with respect to 'x', we get

$$x\frac{d^2y}{dx^2}+1\frac{dy}{dx}+1.\frac{dy}{dx} = Ae^x + Be^{-x} + 2 \Rightarrow x\frac{d^2y}{dx^2}+2\frac{dy}{dx} = xy - x^2 + 2$$

Illustration 5: Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is a general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ **(JEE ADVANCED)**

Sol: Here only one arbitrary constant is present hence we shall differentiate it one time with respect to x and then by substituting the value of c we shall prove the given equation.

Let us find the differential equation for $x^2 - y^2 = c(x^2 + y^2)^2$ ... (i)

Differentiating (i), with respect to 'x', we get $2x - 2y\frac{dy}{dx} = c.2\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right)$ ... (ii)

Substituting the value of c from (i) in (ii), we get

$$\Rightarrow x - y\frac{dy}{dx} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right) \Rightarrow (x^2 + y^2)\left(x - y\frac{dy}{dx}\right) = (x^2 - y^2)\left(2x+2y\frac{dy}{dx}\right)$$

$$\Rightarrow [2y(x^2 - y^2) + y(x^2 + y^2)]\frac{dy}{dx} = x(x^2 + y^2) - 2x(x^2 - y^2) \Rightarrow (3x^2y - y^3)\frac{dy}{dx} = 3xy^2 - x^3$$

$\Rightarrow (x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ As this equation matches the one given in the problem statement. Hence the given equation is the solution for the differential equation.

Hence proved.

Illustration 6: Find the differential equation of the family of curves $y = e^x(a\cos x + b\sin x)$ **(JEE ADVANCED)**

Sol: Since given family of curves have two constants a and b, so we have to differentiate twice with respect to x.

We have, $y = e^x(a\cos x + b\sin x)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x) = y + e^x(-a\sin x + b\cos x)$$

$$\Rightarrow \frac{dy}{dx} - y = e^x(-a\sin x + b\cos x) \qquad \text{... (ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-a\sin x + b\cos x) + e^x(-a\cos x - b\sin x) = \frac{dy}{dx} - y - e^x(a\cos x + b\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - y \qquad [\because e^x(a\cos x + b\sin x) = y] \Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

This is the required differential equation.

Illustration 7: Find the differential equation of all circles which pass through the origin and whose centers lie on the y axis. **(JEE ADVANCED)**

Sol: As circles passes through the origin and whose centers lie on the y axis hence g = 0 and point (0, 0) will satisfy general equation of given circle.

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (i)}$$

Since it passes through origin (0, 0), it will satisfy equation (i)

$$\Rightarrow (0)^2 + (0)^2 + 2g.(0) + 2f.(0) + c = 0 \Rightarrow c = 0$$

$$\Rightarrow x^2 + y^2 + 2gx + 2fy = 0$$

This is the equation of a circle with center (–g, –f) and passing through the origin.

If the center lies on the y-axis, we have g = 0,

$$\Rightarrow x^2 + y^2 + 2.(0).x + 2fy = 0 \Rightarrow x^2 + y^2 + 2fy = 0 \qquad \text{... (ii)}$$

Hence, (ii) represents the required family of circles with center on y axis and passing through origin.

Differentiating (ii) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0 \Rightarrow f = -\left\{\frac{x + y.\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right\}$$

Substituting this value of f in (2), we get

$$x^2 + y^2 - 2y\left(\frac{x + y.\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right) = 0 \Rightarrow (x^2 + y^2)\frac{dy}{dx} - 2xy - 2y^2\left(\frac{dy}{dx}\right) = 0 \Rightarrow (x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

This is the required differential equation.

MASTERJEE CONCEPTS

Curves representing the solution of a differential equation are called integral curves.

Nitish Jhawar (JEE 2009, AIR 7)

6. SOLUTIONS OF DIFFERENTIAL EQUATIONS

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between the dependent and independent variables (free from derivatives) such that it satisfies the given differential equation.

Note: The solution of the differential equation is also called its primitive.

There can be two types of solution to a differential equation:

(a) General solution (or complete integral or complete primitive)

A relation in x and y satisfying a given differential equation and involving exactly the same number of arbitrary constants as the order of the differential equation.

(b) Particular solution

A solution obtained by assigning values to one or more than one arbitrary constant of general solution

Illustration 8: The general solution of $x^2\frac{dy}{dx} = 2$ is **(JEE MAIN)**

Sol: First separate out x term and y term and then integrate it, we shall obtain result.

$\frac{dy}{dx} = \frac{2}{x^2} \Rightarrow dy = \frac{2}{x^2}dx$ Now integrate it. We get $y = -\frac{2}{x} + c$

Illustration 9: Verify that the function $x + y = \tan^{-1}y$ is a solution of the differential equation $y^2y' + y^2 + 1 = 0$ **(JEE MAIN)**

Sol: By differentiating the equation $x + y = \tan^{-1}y$ with respect to x we can prove the given equation.

We have, $x + y = \tan^{-1}y$ ... (i)

Differentiating (i), w.r.t. x we get

$$1 + \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx} \Rightarrow 1 + \frac{dy}{dx}\left(\frac{1+y^2-1}{1+y^2}\right) = 0$$

$$\Rightarrow (1 + y^2) + y^2\frac{dy}{dx} = 0 \Rightarrow y^2y' + y^2 + 1 = 0$$

Illustration 10: Show that the function $y = Ax + \left(2x+2y\frac{dy}{dx}\right)$ is a solution of the differential equation

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ **(JEE MAIN)**

Sol: Differentiating $y = Ax + \frac{B}{x}$ twice with respect to x and eliminating the constant term, we can prove the given equation.

We have, $y = Ax + \frac{dy}{dx} \Rightarrow xy = Ax^2 + B$ ... (i)

Differentiation (i) w.r.t. 'x'. we get $\Rightarrow x\frac{dy}{dx} + 1.y = 2Ax$ ... (ii)

Again differentiating (ii) w.r.t., 'x', we get

$$\Rightarrow x.\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 2A \quad \Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{x\frac{dy}{dx} + y}{x} \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Which is same as the given differential equation. Therefore $y = Ax + \frac{dy}{dx}$ is a solution for the given differential equation.

Illustration 11: If $y.\sqrt{x^2+1} = \log\left[\sqrt{x^2+1}\right]$ show that $(x^2+1)\frac{dy}{dx} + xy + 1 = 0$ **(JEE MAIN)**

Sol: Similar to the problem above, by differentiating $y.\sqrt{x^2+1} = \log\left[\sqrt{x^2+1} - x\right]$ one time with respect to x, we will prove the given equation.

We have, $y.\sqrt{x^2+1} = \log\left[\sqrt{x^2+1}\right]$ ... (i)

Differentiating (i), we get

$$\sqrt{x^2+1}\frac{dy}{dx} + \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}\,y = \frac{(1/2)\left(2x/\sqrt{x^2+1}\right) - 1}{\sqrt{x^2+1} - x} \Rightarrow \sqrt{x^2+1}\frac{dy}{dx} + \frac{x}{\sqrt{x^2+1}} = \frac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}\left[\sqrt{x^2+1} - x\right]};$$

$$(x^2+1)\frac{dy}{dx} + xy = \frac{x - \sqrt{x^2+1}}{\sqrt{x^2+1} - x}; \qquad (x^2+1)\frac{dy}{dx} + xy = -1; \qquad (x^2+1)\frac{dy}{dx} + xy + 1 = 0$$

Illustration 12: Show that y = acos(logx) + bsin(logx) is a solution of the differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$ **(JEE ADVANCED)**

Sol: As the given equation has two arbitrary constants, hence differentiating it two times we can prove it.

We have, y = acos(logx) + bsin(logx) ... (i)

Differentiating (i) w.r.t 'x'. we get ; $\frac{dy}{dx} = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x) \quad \text{... (ii)}$$

Again differentiating with respect to 'x', we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[a\cos(\log x) + b\sin(\log x)] \quad \Rightarrow \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \quad \Rightarrow \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Which is same as the given differential equation

Hence, y = acos(logx) + bsin(logx) is a solution of the given differential equation.

7. METHODS OF SOLVING FIRST ORDER FIRST DEGREE DIFFERENTIAL EQUATION

7.1 Equation of the Form dy/dx = f(x)

To solve this type of differential equations, we integrate both sides to obtain the general solution as discussed below

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad dy = f(x)dx$$

Integrating both sides we obtain $\int dy = \int f(x)dx + c \Rightarrow y = \int f(x)dx + c$

Illustration 13: The general solution of the differential equation $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$ is **(JEE MAIN)**

Sol: General solution of any differential equation is obtained by integrating it hence for given equation we have to integrate it one time to obtain its general equation.

We have: $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$

Integrating, $y = \int\left(x^5 + x^2 - \frac{2}{x}\right)dx + c = \int x^5 dx + \int x^2 dx - 2\int \frac{1}{x}dx + c \Rightarrow y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log|x| + c$

Which is the required general solution.

Illustration 14: The solution of the differential equation $\cos^2 x \frac{d^2y}{dx^2} = 1$ is **(JEE MAIN)**

Sol: By integrating it two times we will get the result.

$\cos^2 x \frac{d^2y}{dx^2} = 1 \Rightarrow \frac{d^2y}{dx^2} = \sec^2 x$

On integrating, we get $\frac{dy}{dx} = \tan x + c_1$

Integrating again, we get $y = \log(\sec x) + c_1 x + c_2$

7.2 Equation of the form dy/dx = f(x) g(y)

To solve this type of differential equation we integrate both sides to obtain the general solution as discussed below

$\frac{dy}{dx} = f(x)g(y) \Rightarrow g(y)^{-1} dy = f(x)dx$

Integrating both sides, we get $\int (g(y))^{-1} dy = \int f(x)dx$

Illustration 15: The solution of the differential equation log(dy/dx) = ax + by is **(JEE MAIN)**

Sol: We can also write the given equation as $\frac{dy}{dx} = e^{ax+by}$. After that by separating the x and y terms and integrating both sides we can get the general equation.

$\frac{dy}{dx} = e^{ax+by} \Rightarrow \frac{dy}{dx} = e^{ax+by} \Rightarrow e^{-by}dy = e^{ax}dx \Rightarrow -\frac{1}{b}e^{-by} = \frac{1}{a}e^{ax} + c$

Illustration 16: The solution of the differential equation $\frac{dy}{dx} = e^{x+y} + x^2 e^y$ is **(JEE MAIN)**

Sol: Here first we have to separate the x and y terms and then by integrating them we can solve the problem above.

The given equation is $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

$\Rightarrow \frac{dy}{dx} = e^x.e^y + x^2 e^y \Rightarrow e^{-y}dy = (e^x + x^2)dx$, Integrating, $\int e^{-y}dy = \int\left(e^x + x^2\right)dx + c$

$\Rightarrow \frac{e^{-y}}{-1} + e^x + \frac{x^3}{3} + c \Rightarrow -\frac{1}{e^y} = e^x + \frac{1}{3}x^3 + c \Rightarrow e^x + \frac{1}{e^y} + \frac{x^3}{3} = C$

7.3 Equation of the Form dy/dx = f (ax+by+c)

To solve this type of differential equation, we put ax + by + c = v and $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dy}{dx} - 0\right)$

$\therefore \frac{dy}{a+bf(v)} = dx$

So solution is by integrating $\int \frac{dy}{a+bf(v)} = \int dx$

Illustration 17: $(x + y)^2 \frac{dy}{dx} = a^2$ **(JEE MAIN)**

Sol: Here we can't separate the x and y terms, therefore put x + y = t hence $\frac{dy}{dx} = \frac{dt}{dx} - 1$. Now we can easily separate the terms and by integrating we will get the required result.

Let x + y = t $\Rightarrow t^2\left(\frac{dt}{dx} - 1\right) = a^2$; $\frac{dt}{dx} = \frac{a^2}{t^2} + 1 = \frac{a^2 + t^2}{t^2}$ $\Rightarrow \int \frac{t^2 dt}{t^2 + a^2} = x + c$

$\Rightarrow \quad t - a\tan^{-1}\frac{dy}{dx} = x + c \Rightarrow \quad y - a\tan^{-1}\frac{x+y}{a} = c$

Illustration 18: $\frac{dx}{dx} = \frac{x+y-1}{\sqrt{x+y+1}}$ **(JEE MAIN)**

Sol: Put x + y +1 = t^2 and then solve similar to the above illustration.

let x + y +1 = t^2

$\Rightarrow \left(2t\frac{dt}{dx} - 1\right) = \frac{t^2 - 2}{t} \quad \Rightarrow \quad \frac{2tdt}{dx} = \frac{t^2 + t - 2}{t} \quad \Rightarrow \quad \int \frac{2t^2}{(t-1)(t+2)} dt = x + c$

$\Rightarrow \quad 2\int\left(1 + \frac{1}{3(t-1)} - \frac{4}{3(t+2)}\right)dt = x + c \quad \Rightarrow \quad 2t + \frac{2\ln|t-1|}{3} - \frac{8\ln|t+2|}{3} = x + c$

$\Rightarrow \quad 2\sqrt{x+y+1} + \frac{2\ln|\sqrt{x+y+1} - 1|}{3} - \frac{8\ln|\sqrt{x+y+1} + 2|}{3} = x + c$

Illustration 19: $\frac{dy}{dx}$ = cos(10x + 8y). Find curve passing through origin in the form y = f(x) satisfying differential equations given **(JEE MAIN)**

Sol: Here first put 10x + 8y = t and then taking integration on both sides we will get the required result.

Let 10x + 8y = t

$\Rightarrow \quad 10 + 8\frac{dy}{dx} = \frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} - 10 = 8\cos t \quad \Rightarrow \quad \int \frac{dt}{8\cos t + 10} \int dx = x + c$

$p = \tan t/2 \quad \frac{dp}{dx} = \frac{1+p^2}{2(1)}\frac{dy}{dx} \Rightarrow \frac{dt}{dx} = \frac{2dp}{1+p^2}$

$\therefore \int \frac{\frac{2dp}{1+p^2}}{8\left(\frac{1-p^2}{1+p^2}\right) + 10} = \int \frac{2dp}{1p^2 + 18} = \int \frac{dp}{p^2 + 9} = x + c$

$\Rightarrow \tan^{-1}(P/3) = x + c \Rightarrow \tan^{-1}\left(\frac{\tan(t/2)}{3}\right) = x + c \Rightarrow 3\tan(x + c) = \tan(10x + 84)$

7.4 Parametric Form

Some differential equations can be solved using parametric forms.

Case I:

$x = r\cos\theta \; y = r\sin\theta$

Squaring and adding $x^2 + y^2 = r^2$ ... (i)

$\tan\theta = \int e^{-y}dy = \int (e^x + x^2)dx + c$ ... (ii)

$xdx + ydy = rdr$ ... (iii)

$\sec^2\theta \, d\theta = \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c \quad \Rightarrow xdy - ydx = x^2\sec^2\theta \, d\theta \quad x = r\cos\theta; \; xdy - ydx = r^2 d\theta$

Case II:

If $x = r\sec\theta, \quad y = r\tan\theta$

$x^2 - y^2 = r^2$ ... (i)

$\frac{1}{e^y} = e^x + \frac{1}{3}x^3 + c = \sin\theta$ ... (ii)

$\Rightarrow \quad xdx - ydy = rdr; \quad xdy - ydx = \cos\theta \, x^2 d\theta \quad \Rightarrow \quad xdy - ydx = r^2\sec\theta d\theta$

Illustration 20: Solve $xdx + ydy = x(xdy - ydx)$ **(JEE MAIN)**

Sol: By substituting $x = r\cos\theta$ and $y = r\sin\theta$ the given equation reduces to $rdr = r\cos\theta(r^2d\theta)$. Hence by separating and integrating both sides we will get the result.

Let $x = r\cos\theta, \; y = r\sin\theta$

Hence the given equation becomes $rdr = r\cos\theta(r^2d\theta)$

$$\int \frac{dr}{r^2} = \int \cos\theta d\theta \quad \Rightarrow \quad -\frac{1}{r} = \sin\theta + c \quad \Rightarrow \quad -\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + c$$

Illustration 21: Solve $\dfrac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\dfrac{1 - x^2 - y^2}{x^2 + y^2}}$ **(JEE ADVANCED)**

Sol: Similar to the problem above, by substituting $x = r\cos\theta$ and $y = r\sin\theta$ the given equation reduces to $\frac{r\,dr}{r^2\,d\theta} = \frac{\sqrt{1-r^2}}{r}$. Hence by integrating both sides we will get the result.

$$\frac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}} \quad \Rightarrow \quad \frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$$

Let $x = r\cos\theta, \; y = r\sin\theta$

$$\frac{rdr}{r^2d\theta} = \frac{\sqrt{1-r^2}}{r} \quad \Rightarrow \quad \int \frac{dr}{\sqrt{1-r^2}} = \theta + c \quad \Rightarrow \quad \sin^{-1}r = \theta + c$$

$$\Rightarrow \quad \sin^{-1}\sqrt{x^2+y^2} = \sin^{-1}\frac{y}{\sqrt{x^2+y^2}} + c$$

Illustration 22: $\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \frac{ydx - xdy}{x}$ **(JEE ADVANCED)**

Sol: Similar to the above illustration.

Let $x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow\ -\frac{rdr}{r^2 d\theta} = \frac{\sqrt{r^2}}{r\cos\theta} \qquad \Rightarrow\ \int \sec\theta\, d\theta + \int \frac{dr}{r} = 0$

$\Rightarrow\ \log(\sec\theta + \tan\theta) + \log r = c \qquad \Rightarrow\ x^2 + y^2 + y\left(\sqrt{x^2 + y^2}\right) + Cx = 0$

7.5 Homogeneous Differential Equations

A differential equation in x and y is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where f(x, y) and g(x, y) are both homogeneous function of the same degree in x and y.

To solve the homogeneous differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$,

substitute y = vx and so $\frac{dy}{dx} = v + x\frac{dy}{dx}$

Thus differential reduces to the form $v + x\frac{dy}{dx} = f(v) \Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v}$

Therefore, solution is $\int \frac{dy}{x} = \int \frac{dv}{f(v) - v} + c$

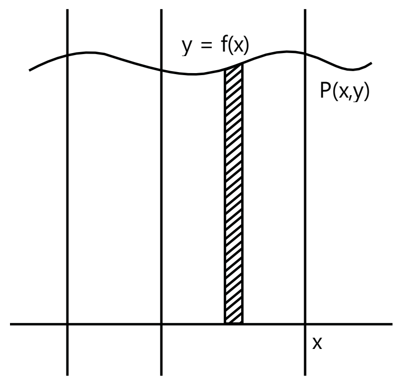


Figure 24.1

Illustration 23: Find the curve passing through (1, 0) such that the area bounded by the curve, x-axis and 2 ordinates, one of which is constant and other is variable, is equal to the ratio of the cube of variable ordinate to variable abscissa. **(JEE MAIN)**

Sol: By differentiating $\int_c^x ydx = \frac{y^3}{x}$, we will get the differential equation.

$A = \int_c^x ydx = \frac{y^3}{x} \qquad \Rightarrow y = \frac{x.3y^2y' - y^3.1}{x^2} \qquad \Rightarrow x^2 = 3xyy' - y^2 \qquad \Rightarrow \frac{dy}{dx} = \frac{x^2 + y^2}{3xy}$

(On differentiating the first integral equation w.r.t x)

Put y = vx; $v + x\frac{dt}{dx} = \frac{1 + v^2}{3v} \quad \Rightarrow \int \frac{3v}{1 - 2v^2} dv = \int \frac{1}{x} dx \quad \Rightarrow -\frac{3}{4}\log\left|1 - 2v^2\right| = \log x + \log c \quad \Rightarrow (x^2 - 2y^2)^3 = cx^2$

Given this curve passes through (1, 0). So, c=1 Hence the equation of curve is $(x^2 - 2y^2)^3 = cx^2$

Illustration 24: The solution of differential equation $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ is **(JEE MAIN)**

Sol: Here by putting y = xv and then integrating both sides we can solve the problem.

Put y = xv $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence the given equation becomes $x\frac{dv}{dx} + v = v + \tan v \Rightarrow x\frac{dv}{dx} = \tan v$

$\Rightarrow\ \frac{dv}{\tan v} = \frac{dx}{x} \ \Rightarrow\ \log \sin v = \log x + \log c \Rightarrow \frac{\sin v}{x} = c \Rightarrow \frac{\sin\left(\frac{y}{x}\right)}{x} = c \Rightarrow cx = \sin\left(\frac{y}{x}\right)$

Illustration 25: Solve $\frac{dy}{dx}=\frac{y^2-2xy-x^2}{y^2+2xy-x^2}$ given y at x = 1 is –1 **(JEE ADVANCED)**

Sol: Similar to the problem above, by putting y = vx, we can solve it and then by applying the given condition we will get the value of c.

Let y = vx

$$\Rightarrow \quad v + x\frac{dv}{dx} = \left(\frac{v^2-2v-1}{v^2+2v-1}\right) \qquad \Rightarrow x\frac{dv}{dx} = -\frac{(v^3+v^2+v+1)}{v^2+2v-1}$$

$$\Rightarrow \quad \int\frac{v^2+2v-1}{(v+1)(v^2+1)}dv = c-\log x \Rightarrow \int\frac{2v(v+1)-(v^2+1)}{(v+1)(v^2+1)}dv = c-\log x$$

$$\Rightarrow \quad \log\left[\frac{(v^2+1)x}{v+1}\right] = \log c \quad \Rightarrow \quad \frac{(v^2-1)x}{(v+1)} = c \Rightarrow \frac{x^2+y^2}{y+x} = c$$

$$\Rightarrow \quad k(x^2+y^2) = x+y$$

Given at x = 1, y = – 1 ⇒ 2k = 0. Hence the required equation is x + y = 0

Illustration 26: Solve $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ given y at x = 1 is $\sqrt{5}$ **(JEE ADVANCED)**

Sol: As we know, when $ax^2+bx+c=0$ then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Hence from given equation $\frac{dy}{dx}=\frac{-2x\pm\sqrt{4x^2+4y^2}}{2y}$ so by putting y = vx and integrating both side, we will get the result.

Given $y\left(\frac{dy}{dx}\right) + 2x\frac{dy}{dx} - y = 0$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{-2x\pm\sqrt{4x^2+4y^2}}{2y} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x\pm\sqrt{x^2+y^2}}{y}$$

Let y = vx

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{\pm\sqrt{v^2+1}-1}{v} - v \Rightarrow x\frac{dv}{dx} = \frac{\pm\sqrt{v^2+1}-1-v^2}{v}$$

$$\Rightarrow \quad \int\frac{vdv}{\pm\sqrt{v^2+1}-(1+v^2)} = \log x + C \quad \Rightarrow \quad \int\frac{vdv}{\pm\sqrt{v^2+1}\left(\mp\sqrt{v^2+1}+1\right)} = \log x + C$$

$$\Rightarrow \quad -\ln\left|\left(\mp\sqrt{v^2+1}+1\right)\right| = \log x + C \quad \Rightarrow \quad x\left|\left(\mp\sqrt{v^2+1}+1\right)\right| = c$$

Given at x = 1, y = v = $\frac{dy}{dx}=\frac{7X-3Y}{-3X+7Y}$ ⇒ $C=\mp\sqrt{6}+1$

$$\Rightarrow \quad \mp\sqrt{y^2+x^2}+x = \mp\sqrt{6}+1$$

This is the required equation.

Note: The obtained solution has 4 equations.

7.6 Differential Equations Reducible to Homogenous Form

A differential equation of the form $\frac{dv}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c}$, where $\frac{a_1}{\theta_2}\neq\frac{b_1}{b_2}$ can be reduced to homogeneous form by adopting the following procedure

Put x = X + h, y = Y + k, so that $\frac{dy}{dx}=\frac{dy}{dx}$

The equation then transforms to $\frac{dY}{dX}=\frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$

Now choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$. Then for these values of h and k the equation becomes

$$\frac{dy}{dx}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$$

This is a homogeneous equation which can be solved by putting Y = vX and then Y and X should be replaced by y – k and x – h.

Special case: If $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$ and $\frac{a}{a'}=\frac{b}{b'}$ = m say, i.e. when coefficient of x and y in numerator and denominator are proportional, then the above equation cannot be solved by the method discussed before because the values of h and k given by the equation will be indeterminate. In order to solve such equations, we proceed as explained in the following example.

Illustration 27: Solve $\frac{dy}{dx}=\frac{3x-6y+7}{x-2y+4}$ **(JEE MAIN)**

Sol: Here the coefficient of x and y in the numerator and denominator are proportional hence by taking 3 common from 3x – 6y and putting x – 2y = v and after that by integrating we will get the result.

$\frac{dy}{dx}=\frac{3x-6y+7}{x-2y+4}=\frac{3(x-2y)+7}{x-2y+4}$; Put x – 2y = v $\Rightarrow 1-2\frac{dy}{dx}=\frac{dy}{dx}$

Now differential equations reduces to $1-\frac{dv}{dx}=2\left(\frac{3v+7}{v+4}\right)$

$$\Rightarrow \frac{dv}{dx}=-5\left(\frac{v+2}{v+4}\right) \qquad \Rightarrow \int\left(1+\frac{2}{v+2}\right)dv=-5\int dx$$

$$\Rightarrow v+2\log|v+2|=-5x+c \qquad \Rightarrow 3x-y+\log|x-2y+2|=c$$

Illustration 28: Solution of differential equation (3y – 7x + 7)dx + (7y – 3x + 3) dy = 0 is **(JEE MAIN)**

Sol: By substituting x = X + h, y = Y + k where (h, k) will satisfy the equation 3y – 7x + 7 = 0 and 7y – 3x + 3 =0 we can reduce the equation and after that by putting Y = VX and integrating we will get required general equation.

The given differential equation is $\frac{dy}{dx}=\frac{7x-3y-7}{-3x+7y+3}$

Substituting x = X + h, y = Y + k, we obtain

$$\frac{dY}{dX}=\frac{(7X-3Y)+(7h-3k-7)}{(-3X+7Y)+(-3h+7k+3)} \qquad \text{... (i)}$$

Choose h and k such that 7h – 3k – 7 = 0 and –3h + 7k + 3 = 0.

This gives h = 1 and k = 0. Under the above transformations, equation (i) can be written as

Let Y = VX so that $\frac{dY}{dX} = V + X\frac{dV}{dX}$, we get $\frac{dY}{dX} = \frac{7X - 3Y}{-3X + 7Y}$

$V + X\frac{dV}{dX} = \frac{-3V+7}{7V-3} \Rightarrow X\frac{dV}{dX} = \frac{7-7V^2}{7V-3} \Rightarrow -7\frac{dX}{X} = \frac{7}{2}.\frac{2V}{V^2-1}dV - \frac{3}{V^2-1}dV$

Integrating, we get

$-7\log X = \frac{7}{2}\log(V^2 - 1) - \frac{3}{2}\log\frac{V-1}{V+1} - \log C \Rightarrow C = (V + 1)^5 (V - 1)^2 X^7 \Rightarrow C = (y + x - 1)^5 (y - x + 1)^2$

Which is the required solution.

7.7 Linear Differential Equation

A differential equation is linear if the dependent variable (y) and its derivative appear only in the first degree. The general form of a linear differential equation of the first order is

$\frac{dy}{dx} + Py = Q$ ..... (i)

where P and Q are either constants or functions of x.

This type of differential equation can be solved when they are multiplied by a factor, which is called integrating factor.

Multiplying both sides of (i) by $e^{\int pdx}$, we get $e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int pdx}$

On integrating both sides with respect to x, we get

$ye^{\int Pdx} = \int Qe^{\int Pdx} + c$ which is the required solution, where c is the constant and $e^{\int pdx}$ is called the integrating factor.

Illustration 29: Solve the following differential equation: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x}$ **(JEE MAIN)**

Sol: We can write the given equation as $e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x}$. By putting e^{-y} = t, we can reduce the equation in the form of $\frac{dt}{dx} + Pt = Q$ hence by using integration factor we can solve the problem above.

We have, $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x} \Rightarrow e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x}$ ... (i)

Put e^{-y} = t. so that $\frac{dy}{dx}$ in equation (i), we get $-\frac{dt}{dx} + \frac{t}{x} = \frac{1}{x} \Rightarrow \frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}$ ... (ii)

This is a linear differential equation in t.

Here, $P = -\frac{1}{x}$ and $Q = -\frac{1}{x}$ ∴ I.F. $= e_{\int Pdx} = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

∴ The solution of (ii) is, t.(I.F.) = $\frac{dy}{dx} = \frac{3x-6y+7}{x-2y+4} = \frac{3(x-2y)+7}{x-2y+4}$

$t\frac{1}{x} = \int\frac{1}{x}\left(-\frac{1}{x}\right)dx + C \Rightarrow \frac{t}{x} = \frac{1}{x} + c \Rightarrow \frac{e^{-y}}{x} = \frac{1}{x} + C$

Illustration 30: The function y(x) satisfy the equation $y(x) + 2x\int_0^x \frac{y(x)}{1+x^2}dx = 3x^2 + 2x + 1$. Prove that the substitution $z(x) = \int_0^x \frac{y(x)}{1 + x^2}dx$ converts the equation into a first order linear differential equation in z(x) and solve the original equation for y(x) **(JEE MAIN)**

Sol: By putting $z'(x) = \frac{y(x)}{1+x^2}$ we will get the linear differential equation in z form and then by applying integrating factor we get the result.

Let $z'(x) = \frac{d(x)}{1+x^2}$ $\Rightarrow$ $z'(x) \times (1 + x^2) + 2x(z(x)) = 3x^2 + 2x + 1$

$$\Rightarrow \frac{dz}{dx} + \frac{2x}{1+x^2}z = \frac{3x^2+2x+1}{x^2+1} \qquad \text{... (i)}$$

This is a first order linear differential equation in z.

$\therefore$ I.F. $= e^{\int Pdx} = e^{\int\frac{2x}{1+x^2}dx} = 1+x^2$ $\therefore$ Solution of (i) is $z(\text{I.F.}) = \int(Q\times \text{I.F})dx + c$

$\Rightarrow z(1 + x^2) = \int \frac{x^3+x^2+x}{x^2+1}(x^2+1)dx + C$ $\Rightarrow z(1 + x^2) = \frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2} + C$ and $y = 3x^2 + 2x + 1 - 2xz$

Illustration 31: Solve the differential equation ysin2x.dx – (1 + y² + cos2x)dy = 0 **(JEE MAIN)**

Sol: Similar to illustration 28, by putting –cos 2x = t, we can reduce the equation in the form of $\frac{dt}{dx}$ + Pt = Q hence by using integration factor we can solve the problem given above.

We have, $y\sin 2x.dx - (1 + y^2 + \cos 2x)dy = 0$

$$\Rightarrow \sin 2x.\frac{dx}{dy} - \frac{\cos 2x}{y} = \frac{1+y^2}{y} \qquad \text{... (i)}$$

Putting –cos 2x = t so that $2\sin 2x\frac{dx}{dy} = \frac{dt}{dy}$ in equation (i), we get $\frac{dt}{dy} + \frac{2}{y}t = 2\left(\frac{1+y^2}{y}\right)$

Here, $P = \frac{2}{y}$ and $Q = 2\frac{1+y^2}{y}$

$\therefore$ I.F. $= e^{\int Pdy} = e^{\int\frac{2}{y}dy} = y^2$ $\therefore$ The solution is t.(I.F.) $= \int(Q\times \text{I.F.})dy + C$

$\Rightarrow$ $t.y^2 = 2\int\frac{1+y^2}{y}.y^2dy = 2\int y + y^3\ dy$ $\Rightarrow$ $t.y^2 = y^2 + \frac{y^4}{2} + C$

On putting the value of t, we get $-.\cos 2x = 1 + \frac{y^2}{2} + Cy^{-2}$

Illustration 32: Solve $y\log y\frac{dx}{dy} + x - \log y = 0$ **(JEE MAIN)**

Sol: By reducing the given equation in the form of $\frac{dx}{dy} + Px = Q$ we can solve this as similar to above illustrations.

We have, $y\log y\frac{dx}{dy} + x - \log y = 0 \Rightarrow \frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$

This is a linear differential equation in x.

Here $P = \frac{1}{y\log y}$, $Q = \frac{1}{y}$; I.F. $= e^{\int\frac{1}{y\log y}dy} = e^{\log(\log y)} = \log y$

The solution is, x(I.F.) = $\int(Q\times I.F.)+C$; xlogy = $\int\frac{1}{y}(\log y)dy+c$ = $\frac{1}{2}(\log y)^2+C$

x = $\frac{1}{2}\log y+C\frac{1}{\log y}$

Illustration 33: Solve $(x + 2y^3)\frac{dx}{dy} = y$ **(JEE ADVANCED)**

Sol: By reducing given equation in the form of $\frac{dx}{dy}$ + Px = Q and then using the integration factor we can solve this.

$(x + 2y^3)\frac{dx}{dy} = y \Rightarrow \frac{dx}{dy} = \frac{x+2y^3}{y} = \frac{x}{y}+2y^2 \Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y^2$

I.F = $e^{-\int\frac{1}{y}dy} = \frac{1}{y}$;

Solutions is $x.\frac{1}{y} = y^2 + C$

Alternate method: $xdy + 2y^3dy = ydx$

$\Rightarrow \quad 2ydy = \frac{ydx-xdy}{y^2} \quad \Rightarrow \quad 2ydy = d\left(\frac{x}{y}\right) \Rightarrow y^2 = \frac{x}{y} + C$

Illustration 34: Let g(x) be a differential function for every real x and g′(0) = 2 and satisfying $g(x+y) = e^y g(x) + 2e^x g(y)$ ∀ x and y. Find g(x) and its range. **(JEE ADVANCED)**

Sol: By using $g'(x) = \lim\limits_{b\to 0}\frac{g(x+h)-g(x)}{h}$ and solving we will get g(x).

$g'(x) = \lim\limits_{b\to 0}\frac{g(x+h)-g(x)}{h}$

$\Rightarrow \quad g'(x) = \lim\limits_{h\to 0}\frac{e^h g(x)+2e^x g(h)-g(x)}{h} \quad \Rightarrow \quad g'(x) = g(x)\lim\limits_{h\to 0}\frac{e^h-1}{h}+2e^x\lim\limits_{h\to 0}\frac{g(h)}{h} \quad \Rightarrow \quad g'(x) = g(x) + 2e^x$

At x = 0, g(x) = 0 ⇒ g(0) = 0

$\frac{dy}{dx} - y = 2e^x \quad \Rightarrow \quad I.F. = e^{-x}$

Solution is $y.e^{-x} = 2x + C$

$g(0) = 0 \Rightarrow C = 0 \Rightarrow g(x) = 2xe^x$

$g'(x) = 2e^x + 2xe^x = 2e^x(x + 1)$

g′(x) = 0 at x = -1; g(–1) = –2/e

$\Rightarrow \quad$ Range of g(x) = $\left[-\frac{2}{e},\infty\right]$

Illustration 35: Find the solution of $(1-x^2)\frac{dy}{dx}+2xy = x\sqrt{1-x^2}$ **(JEE ADVANCED)**

Sol: By reducing given equation in the form of $\frac{dy}{dx}$ + Py = Q and then by using integration factor i.e. $e^{\int Pdx}\left(\frac{dy}{dx}+Py\right) = Qe^{\int pdx}$ we can solve the problem.

$$\frac{dy}{dx} + \frac{2x}{(1-x^2)}y = \frac{x\sqrt{1-x^2}}{1-x^2} \text{ ; I.F. } = e^{\int P dx} = e^{\int \frac{2x}{1-x^2}dx} = \frac{1}{1-x^2}$$

Solution is y. $y\frac{1}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}}\frac{1}{1-x^2}dx + c = \int \frac{x}{(1-x^2)^{3/2}}dx + C = \frac{-1}{2}\int \frac{-2x}{\left(1-x^2\right)^{3/2}}dx + c$

$$y\frac{1}{\left(1-x^2\right)} = \frac{1}{\sqrt{1-x^2}} + c$$

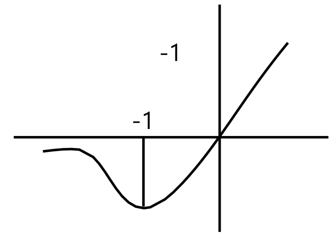


Figure 24.3

MASTERJEE CONCEPTS

Every linear differential equation is of degree 1 but every differential equation of degree 1 is not linear

Shivam Agarwal (JEE 2009, AIR 27)

7.8 Equations Reducible to Linear form

(a) Bernoulli's Equation

A differential equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are function of x and y is called Bernoulli's equation. This form can be reduced to linear form by dividing y^n and then substituting $y^{1-n} = v$

Dividing both sides by y^n, we get, $y^{-n}\frac{dy}{dx} + P.y^{-n+1} = Q$

Putting $y^{-n+1} = v$, so that , $(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$, we get $\frac{dv}{dx} + (1-n)P.y = (1-n)Q$

Which is a linear differential equation

(b) If the given equation is of the form $\frac{dy}{dx} + P.\,f(y) = Q.g(y)$, where P and Q are functions of x alone, we divide the equation by f(y), and then we get $e^{\int P dx} = e^{-\ln(1-x^2)} = \frac{1}{1-x^2}$

Now substitute $\frac{f(y)}{g(y)} = v$ and solve.

Illustration 36: Solve $\frac{dy}{dx} = xy + x^3y^2$ **(JEE MAIN)**

Sol: By rearranging the given equation we will get $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}x = x^3$ and then by putting $\frac{-1}{y} = t$ and using the integration factor we can solve it.

$$\frac{dy}{dx} = xy + x^3y^2 \Rightarrow \frac{dy}{dx} - xy = x^3y^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}x = x^3$$

put $\frac{-1}{y} = t \Rightarrow \frac{dy}{dx} + tx = x^3$

This is a linear differential equation with I.F. $= e^{x^2/2} \Rightarrow t\,e^{x^2/2} = \int e^{x^2/2}x^3dx$

Illustration 37: Find the curve such that the y intercept of the tangent is proportional to the square of ordinate of tangent **(JEE MAIN)**

Sol: Here X = 0 and Y= y – mx i.e. $x\frac{dy}{dx} - y = -ky^2$. Hence by putting $\frac{-1}{y} = 1$ and applying integration factor we will get the result.

$X = 0 \Rightarrow Y = y - mx \Rightarrow x\frac{dy}{dx} - y = -ky^2$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}.\frac{1}{x} = \frac{-k}{x}$

Put $\frac{-1}{y} = t \Rightarrow \frac{dt}{dx} + \frac{t}{x} = \frac{-k}{x}$

$\Rightarrow$ I.f. = x

$\Rightarrow$ Solution is t.x = –kx + C $\Rightarrow \frac{-x}{y} = -kx + C$

7.9 Change of Variable by Suitable Substitution

Following are some examples where we change the variable by substitution.

Illustration 38: Solve $y\sin x\frac{dy}{dx} = \cos x(\sin x - y^2)$ **(JEE MAIN)**

Sol: Here by putting $y^2 = t$, the given equation reduces to $\frac{dt}{dx} + (2\cot x)t = 2\cos x$ and then using the integration factor method we will get result.

$y\sin x\frac{dy}{dx} = \cos x(\sin x - y^2)$

Let $y^2 = t \Rightarrow \frac{1}{2}\sin x\frac{dt}{dx} = \cos x(\sin x - t)$

$\Rightarrow \frac{dt}{dx} = 2\cos x - (2\cot x)t \quad \Rightarrow \frac{dt}{dx} + (2\cot x)t = 2\cos x$

I.F. = $\sin^2 x$

$\Rightarrow$ Solution is $t\sin^2 x = \int 2\cos x.\sin^2 x\,dx$

$y^2\sin^2 x = \frac{2\sin^3 x}{3} + c$

Illustration 39: Solve $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$ **(JEE MAIN)**

Sol: Simply by putting $e^y = t$ and using the integration factor we can solve the above problem.

$\frac{dy}{dx} = e^{x-y}(e^x - e^y) \quad \Rightarrow e^y\frac{dy}{dx} = (e^x)^2 - e^xe^y$

Put $e^y = t \Rightarrow \frac{dy}{dx} + te^x = (e^x)^2$;

I.F. = $e^{\int e dx} = e^{e^x}$

Solution is $te^{e^x} = \int (e^x)^2.e^{e^x}dx$

MASTERJEE CONCEPTS

If we can write the differential equation in the form

$f(f_1(x, y)\, d(f_1(x, y)) + \phi(f_2(x, y) d(f_2(x,y)) + \ldots\ldots = 0$, then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d(x + y) = dx + dy$

(ii) $d(xy) = xdy + ydx$

(iii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$

(vi) $d\left(\frac{y^2}{x}\right) = \frac{xydx - y^2dx}{x^2}$

(vii) $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2dx - 2x^2ydy}{y^4}$

(viii) $d\left(\frac{y^2}{x^2}\right) = \frac{xydx - 2xy^2dx}{x^4}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$

(x) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(xi) $d[\log(xy)] = \frac{xdy + ydx}{xy}$

(xii) $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{xy}$,

(xiii) $d\left[\frac{1}{2}\log\left(x^2 + y^2\right)\right] = \frac{xdx + ydy}{x^2 + y^2}$

(xiv) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$

(xv) $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$

(xvi) $d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$

(xvii) $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy - e^ydx}{x^2}$

(xviii) $d(x^my^n) = x^{m-1}y^{n-1}\,(mydx + nxdy)$

(xix) $d\,\frac{dt}{dx} + \frac{t}{x}$

(xx) $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right) = \frac{xdy - ydx}{x^2 - y^2}$

(xxi) $\frac{d\left[f(x, y)\right]^{1-n}}{1-n} = \frac{f'(x, y)}{\left(f(x, y)\right)^n}$

(xxii) $d\left(\frac{1}{y} - \frac{1}{x}\right) = d\left(\frac{1}{y}\right) - d\left(\frac{1}{x}\right) = \frac{dx}{x^2} - \frac{dy}{y^2}$

Shrikant Nagori (JEE 2009, AIR 30)

8. EXACT DIFFERENTIAL EQUATION

The differential equation Mdx + Ndy = 0, where M and N are functions of x and y, is said to be exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Rule for solving Mdx + Ndy = 0 when it is exact

(a) First integrate the terms in M w.r.t. x treating y as a constant.

(b) Then integrate w.r.t. y only those terms of N which do not contain x.

(c) Now, sum both the above integrals obtained and quote it to a constant i.e. $\int M dx + \int N dy = k$, where k is a constant.

(d) If N has no term which is free from x, the $\int M dx = c$ (y constant)

Following exact differentials must be remembered:

(i) $xdy + ydx = d(xy)$

(ii) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(iv) $\frac{xdy + ydx}{xy} = d(\log xy)$

(v) $\frac{dx + dy}{x + y} = d\log(x + y)$

(vi) $\frac{xdy - ydx}{xy} = d\left(\ell n \frac{y}{x}\right)$

(vii) $\frac{ydx - xdy}{xy} = d\left(\ell n \frac{x}{y}\right)$

(viii) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-2} \frac{y}{x}\right)$

(ix) $\frac{ydy - xdx}{x^2 + y^2} = d\left(\tan^{-1} \frac{y}{x}\right)$

(x) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dy - e^x dy}{y^2}$

9. ORTHOGONAL TRAJECTORY

Definition 1: Two families of curves are such that each curve in either family is orthogonal (whenever they intersect) to every curve in the other family. Each family of curves is orthogonal trajectories of the other. In case the two families are identical then we say that the family is self-orthogonal

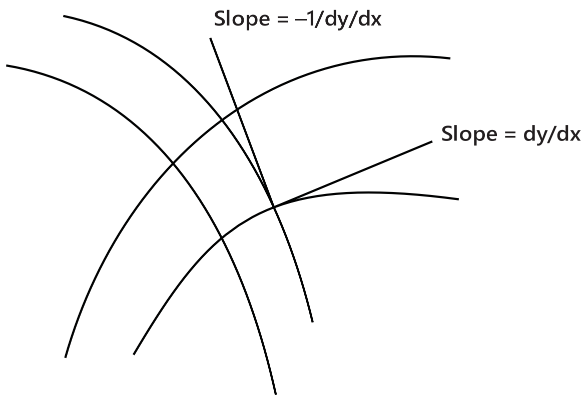


Figure 24.4: Orthogonal trajectories

MASTERJEE CONCEPTS

Orthogonal trajectories have important application in the field of physics. For example, the equipotential lines and the streamlines in an irrotational 2D flow are orthogonal.

Ravi Vooda (JEE 2009, AIR 71)

9.1 How to Find Orthogonal Trajectories

Suppose the first family of curves $F(x, y, c) = 0$ ... (i)

To find the orthogonal trajectories of this family we proceed as follows. First, differentiate (i) w.r.t. x to find $G(x, y, y', c) = 0$ ... (ii)

Now eliminate c between (i) and (ii) to find the differential equation $H(x, y, y') = 0$ ... (iii)

The differential equation for the other family is obtained by replacing y′ with –1/y′. Hence, the differential equation the orthogonal trajectories is H(x, y, –1/y′) = 0 ... (iv)

General solution of (iv) gives the required orthogonal trajectories.

Illustration 40: Find the orthogonal trajectories of a family of straight lines through the origin. **(JEE MAIN)**

Sol: Here as we know, a family of straight lines through the origin is given by y = mx.

Hence by differentiating it with respect to x and eliminating m we will get an ODE of this family and by putting –1/y′ in place of y′ we will get an ODE for the orthogonal family.

The ODE for this family is $xy' - y = 0$

The ODE for the orthogonal family is $x + yy' = 0$

Integrating we find $x^2 + y^2 = c$, which are family of circles with center at the origin.

10. CLAIRAUT'S EQUATION

The differential equation

$y = mx + f(m)$, ... (i)

where $m = \frac{dy}{dx}$ is known as Clairaut's equation.

To solve (i), differentiate it w.r.t. x, which gives

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{df(m)}{dx}$$

$$\Rightarrow \quad x\frac{dm}{dx} + f'(x)\frac{dm}{dx} = 0$$

either $\frac{dm}{dx} = 0 \Rightarrow m = c$ ... (ii)

or $x + f'(x) = 0$ ... (iii)

MASTERJEE CONCEPTS

- If m is eliminated between (i) and (ii),the solution obtained is a general solution of (i)
- If m is eliminated between (i) and (iii), then the solution obtained does not contain any arbitrary constants and is not the particular solution of (i). This solution is called singular solution of (i)

Chinmay S Purandare (JEE 2012, AIR 698)

PROBLEM SOLVING TACTICS

Think briefly about whether you could easily separate the variables or not. Remember that means getting all the x terms (including dx) on one side and all the y terms (including dy) on the other. Don't forget to convert y′ to dy/dx or you might make a mistake.

If it's not easy to separate the variables (usually it isn't) then we can try putting our equation in the form y′ + P(x)y = Q(x).

In other words, put the y′ term and the y term on the left and then you may divide so that the coefficient of y′ is 1. Then we can use the trick of the integrating factor in which we multiply both sides by . $d\left(\frac{e^x}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$. This makes things much simpler, but it's best to see why from doing problems, not from memorizing formulas.

FORMULAE SHEET

(a) Order of differential equation: Order of the highest derivative occurring in the differential equation

(b) Degree of differential equation: Degree of the highest order derivative when differential coefficients are free from radicals and fractions.

(c) General equation : $\frac{dy}{dx} = f(x) \Rightarrow \quad y = \int f(x)dx + c$

(d) $\frac{dy}{dx} = f(ax + by + c)$, then put ax + by + c = v

(e) If $\frac{dy}{dx} = f(x)g(y) \Rightarrow g(y)^{-1}dy = f(x)dx$ then $\int (g(y))^{-1}dy = \int f(x)dx$

(f) Parametric forms

Case I: $x = r\cos\theta$, $y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$; $\tan\theta = \frac{y}{x}$; $xdx + ydy = rdr$; $xdy - ydx = r^2 d\theta$

Case II: $x = r\sec\theta$, $y = r\tan\theta \Rightarrow x^2 - y^2 = r^2$; $\frac{y}{x} = \sin\theta$; $xdx - ydy = rdr$; $xdy - ydx = r^2\sec\theta d\theta$

(g) If $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, then substitute $y = vx \Rightarrow \int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c$

(h) If $\frac{dv}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, then substitute x = X + h, y = Y + k

$$\Rightarrow \frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.

(i) If the equation is in the form of $\frac{dy}{dx} + Py = Q$ then $ye^{\int Pdx} = \int Qe^{\int Pdx} + c$